Modeling the Space of Point Landmark Constrained Diffeomorphisms

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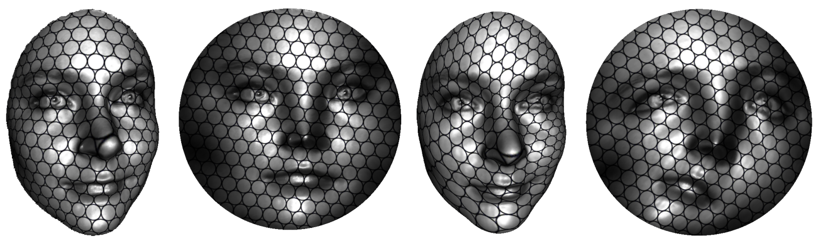
Abstract. Surface registration plays a fundamental role in shape analysis and geometric processing. Generally, there are three criteria in evaluating a surface mapping result: diffeomorphism, small distortion, and feature alignment. To fulfill these requirements, this work proposes a novel model of the space of point landmark constrained diffeomorphisms. Based on Teichmüller theory, this mapping space is generated by the Beltrami coefficients, which are infinitesimally Teichmüller equivalent to 0. These Beltrami coefficients are the solutions to a linear equation group. By using this theoretic model, optimal registrations can be achieved by iterative optimization with linear constraints in the diffeomorphism space, such as harmonic maps and Teichmüller maps, which minimize different types of distortion. The theoretical model is rigorous and has practical value. Our experimental results demonstrate the efficiency and efficacy of the proposed method.

Keywords: Teichmüller Map, Conformal Geometry, Point Landmark Constrained Diffeomorphism

1 Introduction

3D surface registration serves as a fundamental process in shape analysis and geometric processing tasks. In computer vision areas, such as human face registration and tracking [54, 55], human body registration and tracking [16, 3], and general surface registration [34], high-quality surface mappings are desirable. In medical imaging areas, such as brain morphometry [40, 36, 26, 44] and virtual colonoscopy[48, 28], the accuracy of shape classification and abnormality detection relies heavily on the quality of the surface registration results.

In this work, we propose a novel approach to model the space of point landmark constrained diffeomorphisms for 3D surface registration. The generators of this space are the Beltrami coefficients infinitesimally Teichmüller equivalent to 0. This theoretic result can be applied to optimize special energies in the point landmark constrained diffeomorphism space, such as harmonic energy and angle distortion, to obtain harmonic mappings and Teichmüller mappings. The computation of these mappings can be effectively accomplished by solving quadratic optimization problems. As shown in Fig.6, given a male and a female facial surface with landmarks, there are infinite many diffeomorphisms between the two faces with landmark constraints. Conventional Tecihmüller map is only one of them. The current work allows us to perform optimization in this mapping space, for example, a diffeomorphism with landmark constraints with minimal elastic deformation energy (namely generalized harmonic energy).



(a) (b) (c) (d)

Fig. 1: Surface conformal mapping from (a) to (b) and quasi-conformal mapping from (c) to (d).

Criteria for Registrations The following criteria are widely considered effective to evaluate the quality of a surface mapping. *i) Bijection* In most situations, a 1-1 correspondence is desired for surface registration purposes. *ii) Distortion* Surface registration will induce geometric distortions. In applications, it is highly preferred to minimize the distortion. *iii) Feature alignment* Surfaces, such as human faces and human bodies, have natural anatomical features like eye corners, nose tips, and joints. A high-quality surface registration should align these features accurately. *iv) Smoothness* In practice, surface registrations are required to be continuous and even smooth without folding or tearing. For surface registration purposes, an ideal mapping should be smooth, bijective, features aligned, and with least distortion. Thus, we propose an efficient algorithm to find point landmark constrained diffeomorphisms (smooth, bijective) with minimal distortions based on infinitesimal Teichmüller theory.

Space of Diffeomorphisms Based on the quasi-conformal geometry theory [29, 2, 11], the mapping space of all diffeomorphisms between two surfaces is converted to a functional space defined on the source surface.

Consider a pair of Riemannian surfaces $(S, \mathbf{g})$ and $(T, \mathbf{h})$ with the same topology, as shown in the last two frames in Fig. 1, where S is the female face and T is the unit planar disk. A diffeomorphism $f : S \to T$ maps infinitesimal ellipses on the source to infinitesimal circles on the target. The shape of the ellipses (eccentricity and the orientation) is encoded into a complex function, the so-called Beltrami coefficient μ_f. The diffeomorphism f and its Beltrami coefficient μ_f mutually determine each other by the Beltrami equation in Eqn. 4. The space of all diffeomorphisms between the two surfaces is essentially equivalent to the functional space of all Beltrami coefficients, whose norm is less than one almost everywhere (see e.g. [2]).

Point Landmark Constrained Diffeomorphism Space Suppose some landmarks are given $\{p_i\}_{i=1}^n$ on the source surface S and $\{q_j\}_{j=1}^n$ on the target surface T, the landmark matching criteria requires the diffeomorphisms $f : S \to T$ maps each p_i to the corresponding q_i. Using the Beltrami coefficient representation of the mappings, the central question becomes how to choose μ, such that

$$f^\mu(p_i) = q_i,\ \forall i = 1, \dots, n. \tag{1}$$

The diffeomorphisms satisfying the Eqn. 1 form the point landmark constrained diffeomorphism space, denoted as $\mathcal{F}(S \setminus \Gamma, T \setminus \Lambda)$, where $\Gamma = \{p_i\}_{i=1}^n$ and $\Lambda = \{q_j\}_{j=1}^n$. Fig. 2 shows one point landmark constrained diffeomorphism.

Suppose there are two Beltrami coefficients μ_1 and μ_2, such that f^{μ_k} satisfies the point landmark constraints in Eqn. 1, then the composition $(f^{\mu_2})^{-1} \circ f^{\mu_1}$ is an automorphism of the source S, homotopic to identity and fixes all the landmarks p_i's. The point landmark constrained automorphisms form a group

$$G(S \setminus \Gamma) := \{f^\mu : \|\mu\|_\infty < 1,\ f^\mu \sim \mathrm{id}_S,\ f^\mu(p_i) = p_i, \forall i\}, \tag{2}$$

where Γ is the set of landmarks.

$G(S \setminus \Gamma)$ is an infinite dimensional Lie group, we can find a set of its "generators", the so-called infinitesimal Teichmüller trivial diffeomorphisms, $T^0(S \setminus \Gamma)$. Namely, for any $f^\mu \in G(S \setminus \Gamma)$, we can find a sequence of diffeomorphisms f^{μ_i}, $\mu_i \in T^0(S \setminus \Gamma)$, such that

$$\lim_{n\to\infty} f^{\mu_n/n} \circ f^{\mu_{n-1}/n} \circ \cdots f^{\mu_1/n} \to f^\mu,$$

where the space of infinitesimal Teichmüller trivial diffeomorphisms is given by (see e.g. [10])

$$T^0(S \setminus \Gamma) := \left\{\mu : \|\mu\|_\infty < 1,\ \int_S \mu\varphi = 0,\ \forall \varphi \in \Omega(S \setminus \Gamma)\right\}. \tag{3}$$

Thus, the infinitesimal Teichmüller trivial diffeomorphisms $T^0(S\setminus\Gamma)$ generate the point landmark constrained automorphism group $G(S \setminus \Gamma)$. And $G(S \setminus \Gamma)$ gives all the point landmark constrained diffeomorphisms, namely solutions to Eqn. 1.

Optimization in the Space of Diffeomorphisms In practice, the optimal registration can be obtained by searching in the point landmark constrained diffeomorphism space (solutions to Eqn. 1) for a solution that optimizes some specific energy. From the above discussion, the optimization can be carried out within $G(S \setminus \Gamma)$ or $T^0(S \setminus \Gamma)$ instead. Constraints described in Eqn. 3 are linear, and given suitable energy, the optimization will become convex and can be solved by quadratic programming methods.

In this work, we compute point landmark constrained harmonic maps, which minimizes the elastic deformation energy in the point landmark constrained diffeomorphism space. Furthermore, we compute the point landmark constrained Teichmüller map that minimizes the angle distortion, namely the L^∞-norm of the Beltrami coefficient. Our experimental results demonstrate the efficacy and efficiency of the proposed method.

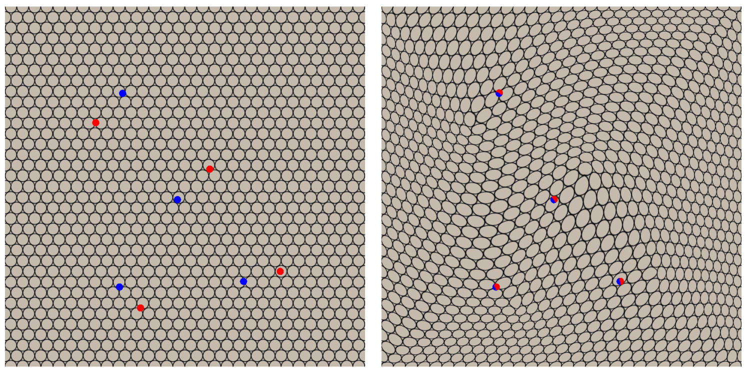


Fig. 2: Feature alignment illustration. The red dots are feature points and the blue dots represent the target position of the feature points. Namely, we are looking for a diffeomorphism that maps the red dots to the blue dots. The bijectivity is visualized by the circle patterns with no overlap or flip. The coincident blue and red dots represent the exact match of feature points.

Contributions The proposed method has the following merits:

- Novel: We propose a novel approach to compute diffeomorphic surface mappings with point landmark constraints. Both point landmark constrained harmonic maps and Teichmüller maps can be computed under the same framework.
- Rigorous: We present a solid mathematical foundation to guarantee the existence of the Teichmüller map which is diffeomorphic, point landmarks aligned, and with least distortion.
- Practical: Based on infinitesimal Teichmüller theory, the feature aligned mapping is computed by iteratively solving quadratic optimization problems.

2 Related Works

3D surface registration methods have been extensively studied in recent years because of their fundamental importance. We list some of the most related work from more than abundant literature and readers are referred to [38, 30, 9, 35] for surveys in surface registration and parameterization.

Point Cloud Registration Point could registration methods can be applied on the vertices of 3D surfaces directly. Iterative Closest Points (ICP) [6, 33] is one of the most well-known methods for point cloud registration. However, ICP is designed to deal with rigid motion situations. To alleviate this issue, multiple researches [4, 5, 14] presented non-rigid ICP methods that focus on non-rigid transformation. However, despite the popularity, these methods fail to guarantee the bijectivity and feature alignment constraints.

Conformal Parameterization Conformal parameterization [13, 39, 49] is a powerful tool in delivering 1-to-1 correspondence between 3D surfaces and 2D domains while preserving local features. Several conformal parameterization registration methods are proposed in 3D facial surface registration and show good results [18, 23, 56, 37]. Many computational approaches have been introduced such as least-square conformal mapping [21, 20], holomorphic differentials based approaches [53] and Ricci flow techniques [18, 17, 49]. However, these conformal parameterization approaches cannot deal with feature constraints while preserving diffeomorphic from surfaces to surfaces. Gu *et al.*[13] proposed to compose an Möbius transformation to the conformal parameterization to minimize landmark mismatch energy. Similar ideas were presented in [42] as well. Lui *et al.*[24, 23] and Choi *et al.*[7] improved this method by optimizing energy functionals consisting of harmonic energy and landmark mismatch energy, however, both approaches failed to either guarantee exact match of the landmarks or retain diffeomorphic given a large number of landmark constraints.

Quasiconformal Mapping Quasiconformal mapping generalizes conformal mapping by allowing bounded angle distortion. Some early quasiconformal mapping algorithms were based on circle packing approaches [15] and were restricted between planar domains [29, 8]. Zeng *et al.*[46, 47] proposed to use the curvature flow approach with auxiliary metrics to compute quasiconformal maps for compact Riemann surfaces. Later, Zeng *et al.*[51] proposed a method to compute quasiconformal maps with curve landmark constraints by integrating quasi-holomorphic 1-forms, and in [52] graph-constrained diffeomorphisms are computed by computing harmonic maps with boundary constraints. To strictly enforce the landmark matching constraints, Lui *et al.*[27] optimized the registration using Beltrami holomorphic flow, where the surface diffeomorphism is obtained by adjusting Beltrami coefficients. Lui *et al.*[25] also applied a similar approach in surface registration compression. To compute general quasiconformal maps between arbitrary Riemann surfaces with similar ideas, Wong *et al.*[45] proposed a vector field approach named discrete Beltrami flow. Even though these approaches perform well in preserving bijectivity, they are less ideal in matching feature points accurately, especially when the shape deformation is large.

Teichmüller Mapping Closely related to quasiconformal maps, Teichmüller map is the quasiconformal map where the L^∞ norm of the Beltrami coefficients is minimized. In other words, the Teichmüller map is the quasiconformal map as close to a conformal map as possible with certain constraints. Weber *et al.*[43] presented an algorithm to approximate the extremal quasiconformal map for genus zero surfaces with boundaries. Lui *et al.*[22] proposed an iterative algorithm by quasi-conformal iteration. A Beltrami holomorphic flow approach to compute Teichmüller extremal map for multiply-connected domains was introduced by Ng *et al.*[31]. In addition to algorithms, Teichmüller space and shape descriptor were applied in surface indexing and classification [19] and medical imaging [41] [50]. For isogeometric analysis purposes, Nian and Chen [32] proposed an iterative algorithm to compute Teichmüller map based on alternating direction method of multipliers.

3 Theoretical Background and Definitions

In the following text, we refer to "point landmark(s)" as "landmark(s)" when there is no confusion.

Beltrami Equation Suppose $f : \mathbb{D} \to \mathbb{D}$ is a complex function, and treated as a mapping from the unit disk $\mathbb{D}$ in the complex plane to itself. The *Beltrami coefficient* μ is defined as

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}. \tag{4}$$

This equation is also called the *Beltrami equation*. The *dilatation* of f is defined as

$$K_f = \frac{1 + |\mu_f|}{1 - |\mu_f|} \tag{5}$$

Then the map f is said to be *quasiconformal* is K_f is bounded, and it's called K-quasiconformal is $K_f \leq K$. A map f is called *conformal* (*holomorphic*), if μ_f is zero everywhere, or equivalently K_f is one everywhere. Intuitively, f maps infinitesimal ellipses to infinitesimal circles, the eccentricity of the ellipse is represented by the ratio between the major axis and the minor axis, which equals to K_f; the angle between the major axis of the ellipse and the horizontal direction is given by $1/2 \arg(\mu)$.

Measurable Riemann Mapping Given a homeomorphism $f : \mathbb{D} \to \mathbb{D}$, its Beltrami coefficient μ_f can be computed by Eqn.4; inversely, given the Beltrami coefficient, there exists a corresponding map.

Theorem 1 (Measurable Riemann Mapping [1]). *Given a measurable complex function $\mu : \mathbb{D} \to \mathbb{C}$, such that $\|\mu\|_\infty < 1$, then there exists a homeomorphism $f : \mathbb{D} \to \mathbb{D}$ satisfying the Beltrami equation 4. Furthermore, two such kind of mappings differ by a Möbius transformation*

$$z \mapsto e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z},$$

where $z_0 \in \mathbb{D}$.

The Beltrami differential is related to the Jacobian of the map f, $J(f)^2 = |f_z|^2(1-|\mu_f|)^2$, hence if $\|\mu_f\|_\infty < 1$, then f is diffeomorphic. This shows the space of all automorphisms of the disk is equivalent to the space of Beltrami coefficients, quotient the Möbius transformation group,

$$\{\text{diffeomorphisms on } \mathbb{D}\} \cong \{\mu | \|\mu\| \infty < 1\}/\{Mobius\}$$

Riemann Surface Suppose S is a topological surface, covered by a set of open sets $S \subset \bigcup U_\alpha$, each set U_α is mapped onto a complex domain $\varphi_\alpha : U_\alpha \to \mathbb{C}$, then $(U_\alpha, \varphi_\alpha)$ is a chart of S, $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas of S. If $U_\alpha \cap U_\beta \neq \emptyset$, then the transition function is given by $\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}$.

Definition 1 (Riemann Surface). *Suppose S is a topological surface with an atlas $\{(U_\alpha, \varphi_\alpha)\}$, if all transition functions are bi-holomorphic, then the atlas is called a conformal atlas, the surface is called a Riemann surface.*

Suppose $(S, \mathbf{g})$ is oriented, then for each point $p \in S$, one can find a small neighborhood U_p such that the isotherm parameterization φ_p exists inside U_p, then all (U_p, φ_p)'s form the conformal atlas, $(S, \mathbf{g})$ is a Riemann surface.

Holomorphic Quadratic Differential Suppose S is a Riemann surface with a conformal atlas $\{(U_i, z_i)\}$, where z_i is the isothermal parameter inside U_i.

Definition 2 (Holomorphic Quadratic Differential). *A holomorphic quadratic differential on a Riemann surface S is an assignment of a function $\phi_i(z_i)$ on each local chart z_i such that if z_j is another local coordinate, we have*

$$\phi_i(z_i) = \phi_j(z_j)\left(\frac{dz_j}{dz_i}\right)^2.$$

We denote the space of all holomorphic differentials on S as $\Omega(S)$. Given a holomorphic quadratic differential $\varphi \in \Omega(S)$, a curve γ is called the horizontal trajectory of φ, if the integration of $\sqrt{\varphi}$ along γ is always a real number. Fig. 3 illustrates the horizontal trajectories of holomorphic quadratic differentials on the cat surfaces.

$\Omega(S)$ is a complex linear space. For a genus $g > 1$ closed Riemann surface S, $\Omega(S)$ is $3g-3$ dimensional. If S is a sphere with n punctures,

$$S = \mathbb{C} \cup \{\infty\} - \{a_1, a_2, \cdots, a_n\}$$

then every holomorphic quadratic differential has the form $\varphi(z)dz^2$, where

$$\phi(z) = \sum_{k=1}^{n} \frac{\rho_k}{z - a_k},$$

such that

$$\sum_{k=1}^{n} \rho_k = 0,\ \sum_{k=1}^{k} \rho_k a_k = 0,\ \sum_{k=1}^{n} \rho_k a_k^2 = 0.$$

Fig. 3: Holomorphic quadratic differentials.

For $S = \mathbb{D}\backslash\{z_1, z_2, ..., z_n\}$,

$$\phi_k(z) = \frac{\eta}{(z - z_k)}, 1 \leq k \leq n \tag{6}$$

form a basis of n dimensional complex vector space, where η is a constant such that $\|\phi\| = \int_S |\phi| = 1$

Beltrami Differential Given a diffeomorphism between two Riemann surfaces $f : (S_1, \{z_i\}) \to (S_2, \{w_j\})$, the Beltrami differential can be defined as

$$\frac{\partial w_j}{\partial \bar{z}_i} d\bar{z}_i = \mu(z_i) \frac{\partial w_j}{\partial z_i} dz_i.$$

Then Beltrami differential $\mu(z_i)d\bar{z}_i/dz_i$ is invariant under the transition maps and thus is globally defined. The K-quasiconformal map can be generalized to the Riemann surface cases directly.

Teichmüller Equivalence We are interested in such kind of homeomorphisms that fix the landmarks.

Definition 3 (Landmark Preserving Automorphism). *Suppose S is a Riemann surface, with landmarks $\Gamma = \{p_1, p_2, \cdots, p_n\}$, $f : S \to S$ is a diffeomorphism homotopic to the identity map, preserving the landmarks, $f(p_i) = p_i$, $i = 1, 2, \ldots, n$, then we say f is a landmark preserving automorphism.*

All the landmark preserving automorphisms form a group, denoted as $G(S \setminus \Gamma)$.

Definition 4 (Teichmüller Trivial). *Suppose μ is a Beltrami differential for a Riemann surface with landmarks, if f^μ is a landmark preserving automorphism, then μ is called Teichmüller equivalent to 0, or Teichmüller trivial, denoted as $\mu \sim 0$.*

The group of landmark preserving automorphism is isomorphic to the space of Teichmüller trivial Beltrami differentials,

$$G(S \setminus \Gamma) \cong \{\|\mu\|_\infty < 1, \mu \sim 0\}.$$

In practice, it is difficult to compute Teichmüller trivial Beltrami differential directly. Instead, we seek for infinitesimally teichmüller trivial differentials.

Definition 5 (Infinitesimal Teichmüller Equivalence). *Two Beltrami differentials μ and ν are called infinitesimally equivalent if $\forall \phi \in \Omega(S)$ with $\|\phi\| = 1$,*

$$\int_S \mu\phi = \int_S \nu\phi \tag{7}$$

The space of Beltrami differentials infinitesimally Teichmüller equivalent to ν is given by

$$T^\nu(S) := \left\{ \mu : \|\mu\|_\infty < 1, \int_S (\mu - \nu)\phi = 0, \forall \phi \in \Omega(S) \right\}. \tag{8}$$

Geometrically, if $\mu \in T^\nu(S)$ is infinitesimally Teichmüller equivalent to ν, then when $t \to 0$

$$f^{\nu + t(\mu - \nu)}(p_i) = f^\nu(p_i) + o(t), \ \forall p_i \in \Gamma.$$

Teichmüller Map In general cases, the Teichmüller map is the one that minimizes the angle distortion.

Definition 6 (Extremal Map). *Let $f : S_1 \to S_2$ be a quasiconformal map between S_1 and S_2. f is said to be an extremal mapping if for any quasiconformal map $h : S_1 \to S_2$ isotopic to f relative to the boundary,*

$$K(f) \leq K(h) \tag{9}$$

where $K(f) = (1 + ||\mu||_\infty)/(1 - ||\mu||_\infty)$ is the maximal dilation. It is uniquely extremal if the inequality in Eqn.9 is strict when $h \neq f$.

Definition 7 (Teichmüller Map). *Let $f : S_1 \to S_2$ be a quasiconformal map. f is said to be a Teichmüller map associated to the holomorphic quadratic differential ϕdz^2, if its associated Beltrami differential is of the form*

$$\mu(f) = k \frac{\bar{\phi}}{|\phi|} \tag{10}$$

for some constant $k < 1$ and quadratic differential with $\|\phi\| < \infty$.

Under general conditions, the Teichmüller map is the extremal quasiconformal map within its homotopy class.

4 Algorithm

Based on previous theories, optimization in the space of landmark constrained diffeomorphisms can be simplified to the optimization in the space of infinitesimally equivalent Beltrami coefficients.

In this section, we first present the general procedure of optimization in the space of infinitesimally equivalent Beltrami coefficients. Then we present an algorithm to compute Teichmüller map as a showcase of this general procedure.

Given $f_0 : \mathbb{D} \to \mathbb{D}$, with constraints $f_0(p_i) = q_i, p_i \in \Gamma$. The Beltrami coefficient of f_0 is μ_0, the problem we consider is

$$\begin{aligned} \min \quad & E(f,\mu) \\ s.t. \ & \int_{\mathbb{D}} \mu\phi_j = \int_{\mathbb{D}} \mu_0\phi_j, \forall\ \phi_j \in \Omega(\mathbb{D}) \end{aligned} \tag{11}$$

where $\mu(z) = f_{\bar{z}}/f_z$.

In discrete setting, the domain $\mathbb{D}$ is represented as a triangle mesh $D = \bigcup \triangle_i$. The Beltrami coefficient is represented as a piecewise constant function on the triangle mesh $\mu = \sum \mu_i \triangle_i = (\mu_1, \mu_2, \cdots, \mu_T)^T$. The infinitesimal equivalence condition can be discretized as

$$\sum_i \mu_i a_{ji} = \sum_i \mu_{0(i)} a_{ji}$$

where $a_{ji} = \int_{\triangle_i} \phi_j$. These are linear constraints on μ.

Beside the constraints in 11, it's usually desirable to add more constraints in order to ensure μ as well as corresponding f have desired properties. For example, $|\mu| < 1$ is a common constraint to add to ensure the resulting map to be bijective. The energy E can have various forms depending on the diffeomorphism we want. Usually, we derive the energy form E based on properties of corresponding map f, which is the most challenging part of our algorithm.

The optimization problem can be solved iteratively. From initial μ_0, we solve the minimization problem 11 with either linear programming or quadratic programming to obtain $\nu = \arg\min_\mu E(f,\mu)$. Next, we have to ensure ν is indeed Beltrami coefficient of some f. We can solve the Beltrami equation to obtain f, or in some cases obtain f by closed form [12]. In either way, the f we obtained may slightly move the landmark. We can diffuse f locally to restore landmark constraints. Then we solve the minimization problem again to obtain a new ν. This procedure is performed iteratively until the optimal μ is attained as well as the optimal diffeomorphism f. Based on our experiments, this iterative procedure usually converges in a few iterations. For the Teichmüller problem we will introduce later, it's proved that the iterative procedure converges to a given precision in finite steps [12].

If landmarks and their targets are too far away, it's advisable to move landmarks to targets gradually. For a sequence of initial map $\{f_0^t, t = 1, 2, \cdots, T\}$, $f_0^t(p_i) = \frac{1-t}{T} p_i + \frac{t}{T} q_i$. We apply above optimization procedure for f_0^t to obtain f^t, then use f^t to initialize f_0^{t+1} by diffusing landmarks to next positions.

The algorithm is summarised as in Algorithm.1:

In figure 4, we illustrate the optimization within the space of landmark preserving diffeomorphisms. The initial map f_0 takes left to the middle with landmark constraints. For the map f from left to right, its Beltrami coefficient μ is infinitesimal equivalent to μ_0 of f_0. The landmark constraints are preserved.

Algorithm 1: Optimization in infinitesimal equivalence space

```
Result: Optimized μ and f
take some f_0 and μ^0;
while stop criteria is not satisfied do
    solve problem 11 to obtain μ;
    solve Beltrami equation 4 to obtain f;
    restore landmark constraints for f;
    compute μ = f_z̄/f_z;
    let μ^0 = μ;
end
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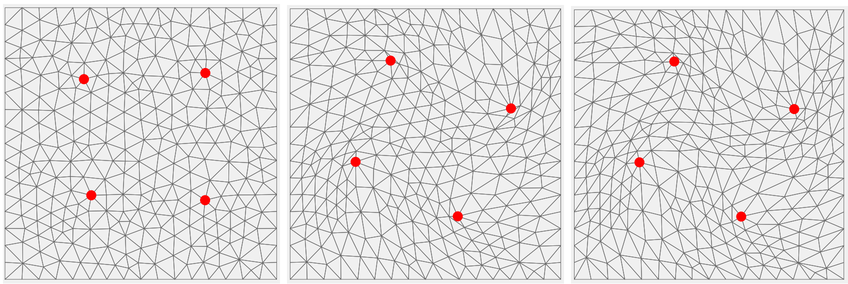


Fig. 4: Infinitesimal equivalence

4.1 Harmonic map

Let $f : \mathbb{D} \to \mathbb{D}$ be a map that takes some feature points $\Gamma = \{p_1, p_2, \cdots, p_k\}$ to some target locations $Q = \{q_1, q_2, \cdots, q_k\}$, The harmonic energy of f can be defined as

$$E(f) = \int_{\mathbb{D}} (|f_z|^2 + |f_{\bar{z}}|^2) dz d\bar{z}$$

Without landmark constraints, by variational principle, the Euler–Lagrange equation for $E(f)$ is

$$\Delta f = 0 \tag{12}$$

This Laplace equation can be solved together with some boundary conditions. However with landmark constraints, if we simply enforce those landmark constraints, the solution to 12 generally is not diffeomorphic. The harmonic map with landmark constraints can be solved in the proposed space of landmark preserving diffeomorphisms.

Since $f_{\bar{z}} = \mu f_z$, we obtain

$$E(f, \mu) = \int_{\mathbb{D}} |f_z|^2 (1 + |\mu|^2) dz d\bar{z}$$

The map f is represented as a piecewise linear function, thus f_z on each triangle $\triangle_i$ is constant and so is μ. So the energy can be integrated as

$$E(f,\mu) = \sum_i |f_z|_i^2 (1 + |\mu_i|^2) A_i$$

where A_i is the triangle area of triangle $\triangle_i$. From initial map f, we can compute f_z and μ, then we optimize μ and f using algorithm 1. Figure 5 shows an initial map with landmark constraints and harmonic map obtained from algorithm 1.

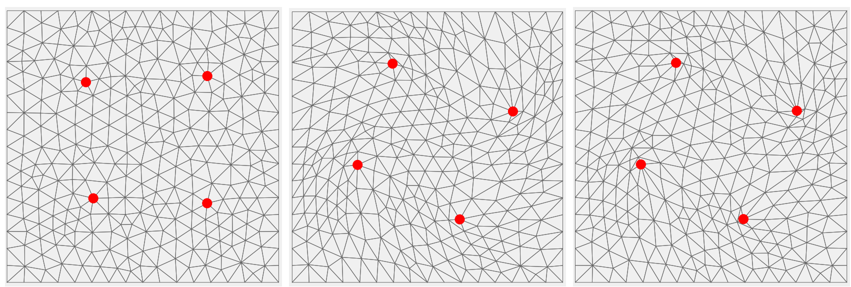


Fig. 5: Harmonic map. Left is a surface with landmarks. The initial map takes left to middle with landmark constraints. The harmonic map is from left to right.

4.2 Teichmüller map

We apply previous general procedure to compute Teichmüller map via infinitesimal approach. Let $\mathbb{D}$ be extended complex plane or unit disk. Given an initial map f_0, which takes some feature points $\Gamma = \{p_1, p_2, \cdots, p_k\}$ to some target locations $Q = \{q_1, q_2, \cdots, q_k\}$, we find the Teichmüller map in the homotopy class of f_0 which preserves the feature points.

The quadratic differentials on $\mathbb{D}$ have closed form 6. Based on Teichmüller theory, we should minimize the L_∞ norm of μ.

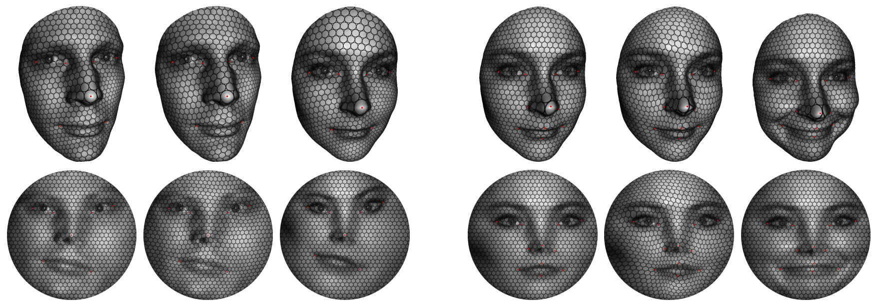
$$\begin{aligned} &\min E(f,\mu) = \|\mu\|_\infty \\ &s.t. \textstyle\int_{\mathbb{D}} \mu\phi_i = \int_{\mathbb{D}} \mu_0 \phi_i, \forall\ \phi_i \in \Omega(\mathbb{D}, \Gamma) \end{aligned} \tag{13}$$

Quadratic differential $\phi_j(z)$ has a simple pole at p_j and analytic elsewhere, so it's integrable on every triangle and on $\mathbb{D}$. We denote the integral as

$$A = (a_{ji}) = \int_{\triangle_i} \phi_j \tag{14}$$

We further separate the real part and imaginary part and obtain

$$\begin{aligned} A_r x - A_i y = b_r \\ A_i x + A_i y = b_i \end{aligned} \tag{15}$$



(a) (b)

Fig. 6: The Teichmüller map between two human face surfaces (a) and the Teichmüller map between human face surfaces with neutral and smiling expressions (b).

where $\mu_i = x_i + iy_i, A = A_r + iA_i, b = b_r + ib_i = A\mu_0$. The L_∞ minimization problem can be solved by introducing an auxiliary variable z with constraints $|\mu_i|^2 = x_i^2 + y_i^2 \leq z$. So we get a equivalent minimization problem

$$\begin{aligned} \min \quad & z \\ s.t \quad & x_i^2 + y_i^2 \ < \ z \\ & A_r x - A_i y = b_r \\ & A_i x + A_i y = b_i \end{aligned} \tag{16}$$

which is a linear programming problem with quadratic constraints. It can be efficiently solved using e.g., interior point method.

5 Experiment Results

We show the optimized diffeomorphisms between 3D human faces. We manually select a few landmarks at the nose, eye corner and mouth corner to ensure meaningful matching. For 3D surfaces, we first conformally map them to the 2D unit disk. Then we compute the Teichmüller map mapping the unit disk to itself with prescribed landmark constraints. Fig. 6(a) and Fig. 6(b) show the Teichmüller maps. In the two figures, the left and right columns are original surfaces textured with circle patterns. In the middle columns, we draw the ellipses which are deformed from each circle by the mappings. Note that in either case, all ellipses have the same eccentricities since Teichmüller map has the property of having the same $|\mu|$ almost everywhere (see Eqn. 10).

In Fig.7 we plot the L^∞-norm of μ corresponding to the initial maps and the optimized Teichmüller maps. We observe that the initial mappings have large distortions near landmark constraints, while in the Teichmüller maps the distortion is uniformly smoothed.

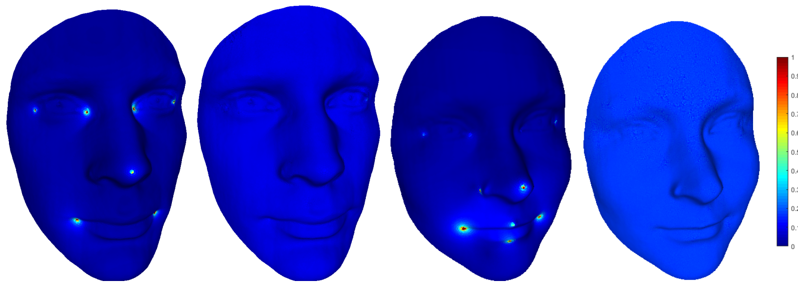


Fig. 7: Heat map of L^∞-norm of μ. The first and third columns correspond to initial mappings and the second and fourth correspond to the optimized Teichmüller maps.

Our experiments are conducted on a desktop computer with Intel i7 4.3GHz CPU. The human face surfaces in Fig. 6(a) and Fig. 6(b) have 40K triangles. The optimizations converge within 20 iterations and take 32s. The convergence of optimization is shown in Fig. 8.

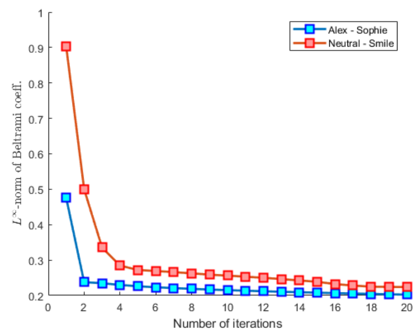


Fig. 8: Convergence of the optimization.

6 Conclusion

This work proposes a model of the space of diffeomorphisms with landmark constraints. The generators of the space consists of infinitesimal Teichmüller trival Beltrami coefficients. The harmonic map and Teichmüller maps can be obtained by solving convex optimizations with quadratic programming. The method is rigorous and practical. The experimental results demonstrate the efficacy and efficiency of the proposed method.

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