A Consistently Fast and Globally Convergent Solution to the Perspective-n-Point Problem

Supplementary material for ECCV 2020 paper 1969

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1 Invertibility of matrix $\sum_{i=1}^{n} \boldsymbol{Q}_i$

Proposition 1. *Unless all projections have the same coordinates, the matrix $\sum_{i=1}^{n} \boldsymbol{Q}_i$ is always invertible.*

Proof. Let $\boldsymbol{m}_i = \begin{bmatrix} x_i & y_i & 1 \end{bmatrix}^T$ be the projection of point $\boldsymbol{M}_i$ on the Euclidean plane Z=1. Then, matrix Q_i is

$$\boldsymbol{Q}_i = \left(\boldsymbol{m}_i \boldsymbol{1}_z^T - \boldsymbol{I}_3\right)^T \left(\boldsymbol{m}_i \boldsymbol{1}_z^T - \boldsymbol{I}_3\right) = \begin{bmatrix} 1 & 0 & -x_i \\ 0 & 1 & -y_i \\ -x_i & -y_i & x_i^2 + y_i^2 \end{bmatrix} \tag{1}$$

Summing up eq. (1) over all points, we have:

$$\sum_{i=1}^{n} \boldsymbol{Q}_i = \begin{bmatrix} n & 0 & -\sum_{i=1}^{n} x_i \\ 0 & n & -\sum_{i=1}^{n} y_i \\ -\sum_{i=1}^{n} x_i & -\sum_{i=1}^{n} y_i & \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 \end{bmatrix}. \tag{2}$$

The determinant of $\sum_{i=1}^{n} \boldsymbol{Q}_i$ expands as

$$\det\left(\sum_{i=1}^{n} \boldsymbol{Q}_i\right) = n^3 \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right) + n^3 \left(\frac{1}{n}\sum_{i=1}^{n} y_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)^2\right), \tag{3}$$

which is the sum of the variances of x_i and y_i scaled by n^3. Since the only case in which the variance vanishes is when the data are identical, the matrix $\sum_{i=1}^{n} \boldsymbol{Q}_i$ will be invertible unless all projections have the same coordinates. □

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2 Proper orthonormality constraints

A 3×3 matrix represents a proper rotation if it satisfies the orthonormality constraints (i.e., unit norm and mutual orthogonality) and the additional requirement of a positive unit determinant. The proper orthonormality function $\boldsymbol{h} : \mathbb{R}^9 \rightarrow \mathbb{R}^6$ incorporates the aforementioned constraints and is defined so that $\boldsymbol{x} \in \mathbb{R}^9$ corresponds to a valid rotation matrix when $\boldsymbol{h}(\boldsymbol{x}) = \mathbf{0}$:

$$\boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{x}_{1:3}^T \boldsymbol{x}_{1:3} - 1 \\ \boldsymbol{x}_{4:6}^T \boldsymbol{x}_{4:6} - 1 \\ \boldsymbol{x}_{1:3}^T \boldsymbol{x}_{4:6} \\ \boldsymbol{x}_{1:3}^T \boldsymbol{x}_{7:9} \\ \boldsymbol{x}_{4:6}^T \boldsymbol{x}_{7:9} \\ \det\left(\mathrm{mat}(\boldsymbol{x})\right) - 1 \end{bmatrix} \tag{4}$$

The notation $\boldsymbol{x}_{i:j}$ denotes the subvector of $\boldsymbol{x}$ consisting of components x_k with $i \leq k \leq j$. The unit norm constraint associated with $\boldsymbol{x}_{7:9}$ is redundant and hence omitted from $\boldsymbol{h}$. The Jacobian $\boldsymbol{H}_{\mathbf{x}}$ of the proper orthonormality constraints at $\boldsymbol{x} = \mathbf{x}$ is

$$\boldsymbol{H}_{\mathbf{x}} = \left.\frac{\partial \boldsymbol{h}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\mathbf{x}} = \begin{bmatrix} \mathbf{x}_{1:3}^T & \mathbf{0}_3^T & \mathbf{0}_3^T \\ \mathbf{0}_3^T & \mathbf{x}_{4:6}^T & \mathbf{0}_3^T \\ \mathbf{x}_{4:6}^T & \mathbf{x}_{1:3}^T & \mathbf{0}_3^T \\ \mathbf{x}_{7:9}^T & \mathbf{0}_3^T & \mathbf{x}_{1:3}^T \\ \mathbf{0}_3^T & \mathbf{x}_{7:9}^T & \mathbf{x}_{4:6}^T \\ \bar{\mathbf{x}}_{1:3}^T & \bar{\mathbf{x}}_{4:6}^T & \bar{\mathbf{x}}_{7:9}^T \end{bmatrix}, \tag{5}$$

where the matrix representation of $\mathbf{x}$ is

$$\mathbf{X} = \mathrm{mat}(\mathbf{x}) = \begin{bmatrix} \mathbf{x}_{1:3}^T \\ \mathbf{x}_{4:6}^T \\ \mathbf{x}_{7:9}^T \end{bmatrix} = \begin{bmatrix} \mathrm{x}_{11} & \mathrm{x}_{12} & x_{13} \\ \mathrm{x}_{21} & \mathrm{x}_{22} & \mathrm{x}_{23} \\ \mathrm{x}_{31} & \mathrm{x}_{32} & \mathrm{x}_{33} \end{bmatrix} \tag{6}$$

and $\bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{x}}_1^T & \bar{\mathbf{x}}_2^T & \bar{\mathbf{x}}_3^T \end{bmatrix}^T$ is the vector representation of the transposed adjoint of the matrix represented by $\mathbf{x}$:

$$\mathrm{mat}(\bar{\mathbf{x}}) = \mathrm{adj}\left(\mathbf{X}\right)^T = \begin{bmatrix} \mathrm{x}_{22}\mathrm{x}_{33} - \mathrm{x}_{23}\mathrm{x}_{32} & \mathrm{x}_{23}\mathrm{x}_{31} - \mathrm{x}_{21}\mathrm{x}_{33} & \mathrm{x}_{21}\mathrm{x}_{32} - \mathrm{x}_{22}\mathrm{x}_{31} \\ \mathrm{x}_{13}\mathrm{x}_{32} - \mathrm{x}_{12}\mathrm{x}_{33} & \mathrm{x}_{11}\mathrm{x}_{33} - \mathrm{x}_{13}\mathrm{x}_{31} & \mathrm{x}_{12}\mathrm{x}_{31} - \mathrm{x}_{11}\mathrm{x}_{32} \\ \mathrm{x}_{12}\mathrm{x}_{23} - \mathrm{x}_{13}\mathrm{x}_{22} & \mathrm{x}_{13}\mathrm{x}_{21} - \mathrm{x}_{11}\mathrm{x}_{23} & \mathrm{x}_{11}\mathrm{x}_{22} - \mathrm{x}_{12}\mathrm{x}_{r21} \end{bmatrix}. \tag{7}$$

3 Rank of the Jacobian of the proper orthonormality constraints

Proposition 2. *Let* $\boldsymbol{H}_{\mathbf{x}} \equiv \left.\frac{\partial \boldsymbol{h}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\mathbf{x}} \in \mathbb{R}^{6\times 9}$ *be the Jacobian matrix of the proper orthonormality function at* $\mathbf{x}$*. If* $\mathrm{rank}\left(\mathbf{X}\right) \geq 2$*, then* $\mathrm{rank}\left(\boldsymbol{H}_{\mathbf{x}}\right) = 6$.

Proof. To prove that the rank of $\boldsymbol{H}_{\mathbf{x}}$ is 6, we resort to its reduced echelon form. Since $\mathrm{rank}\left(\mathbf{X}\right) \geq 2$, there will be at least two non-vanishing minors of $\mathbf{X}$. Without loss of generality, we assume that the minor determinant corresponding to element x_{13} is not zero. Then, at least two of the 4 elements in the minor should be non zero. Again, without loss of generality, we assume $\mathrm{x}_{31} \neq 0$ and obtain the row echelon form of $\boldsymbol{H}_{\mathbf{x}}$:

$$\operatorname{Rref}(\boldsymbol{H}_{\mathbf{x}}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{\mathrm{x}_{12}\mathrm{x}_{33}-\mathrm{x}_{13}\mathrm{x}_{32}}{d_{13}} & 0 & \frac{\mathrm{x}_{12}}{\mathrm{x}_{31}} & \frac{\mathrm{x}_{12}\mathrm{x}_{21}\mathrm{x}_{33}-\mathrm{x}_{13}\mathrm{x}_{22}\mathrm{x}_{31}}{\mathrm{x}_{31}d_{13}} \\ 0 & 1 & 0 & 0 & 0 & \frac{\mathrm{x}_{11}\mathrm{x}_{33}-\mathrm{x}_{13}\mathrm{x}_{31}}{d_{13}} & 0 & -\frac{\mathrm{x}_{11}}{\mathrm{x}_{31}} & -\frac{\mathrm{x}_{21}(\mathrm{x}_{11}\mathrm{x}_{33}-\mathrm{x}_{13}\mathrm{x}_{31})}{\mathrm{x}_{31}d_{13}} \\ 0 & 0 & 1 & 0 & 0 & -\frac{\mathrm{x}_{11}\mathrm{x}_{32}-\mathrm{x}_{12}\mathrm{x}_{31}}{d_{13}} & 0 & 0 & \frac{\mathrm{x}_{11}\mathrm{x}_{22}-\mathrm{x}_{12}\mathrm{x}_{21}}{d_{13}} \\ 0 & 0 & 0 & 1 & 0 & -\frac{\mathrm{x}_{22}\mathrm{x}_{33}-\mathrm{x}_{23}\mathrm{x}_{32}}{d_{13}} & 0 & \frac{\mathrm{x}_{22}}{\mathrm{x}_{31}} & \frac{\mathrm{x}_{22}(\mathrm{x}_{21}\mathrm{x}_{33}-\mathrm{x}_{23}\mathrm{x}_{31})}{\mathrm{x}_{31}d_{13}} \\ 0 & 0 & 0 & 0 & 1 & \frac{\mathrm{x}_{21}\mathrm{x}_{33}-\mathrm{x}_{23}\mathrm{x}_{31}}{d_{13}} & 0 & -\frac{\mathrm{x}_{21}}{\mathrm{x}_{31}} & -\frac{\mathrm{x}_{21}(\mathrm{x}_{21}\mathrm{x}_{33}-\mathrm{x}_{23}\mathrm{x}_{31})}{\mathrm{x}_{31}d_{13}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{\mathrm{x}_{32}}{\mathrm{x}_{31}} & \frac{\mathrm{x}_{33}}{\mathrm{x}_{31}} \end{bmatrix}, \tag{8}$$

where d_{13} is the minor

$$d_{13} = \det \begin{bmatrix} \mathrm{x}_{21} & \mathrm{x}_{22} \\ \mathrm{x}_{31} & \mathrm{x}_{32} \end{bmatrix}. \tag{9}$$

By definition, the rows of $\operatorname{Rref}(\boldsymbol{H}_{\mathbf{x}})$ are linearly independent, hence $\operatorname{rank}(\boldsymbol{H}_{\mathbf{x}}) = 6$.

Suppose now that $\mathrm{x}_{31} = 0$. Then, since $d_{13} \neq 0$, it follows that $\mathrm{x}_{21} \neq 0$ and $\mathrm{x}_{32} \neq 0$. Since rearranging the rows of $\mathbf{X}$ does not affect its rank, we may swap the second row with the third, thereby reverting back to the case where $\mathrm{x}_{31} \neq 0$ that yields $\operatorname{rank}(\boldsymbol{H}_{\mathbf{x}}) = 6$.

□

4 Angle between any point on the 8-sphere of radius $\sqrt{3}$ and its closest rotation matrix

Proposition 4. *Let $\boldsymbol{e} \in \mathbb{R}^9$ such that $\|\boldsymbol{e}\| = \sqrt{3}$. If the vector $\boldsymbol{r}$ represents a rotation matrix that minimizes the Frobenius distance from $\boldsymbol{e}$, then the angle between $\boldsymbol{e}$ and $\boldsymbol{r}$ is strictly less than $71°$.*

Proof. Finding the orthogonal matrix that minimizes the Frobenius distance from a given matrix is known as the nearest orthogonal approximation problem [1], and is closely related to the absolute orientation problem [4]. For the special case of rotation matrices, it was shown by Horn et al. in [3] that the rotation matrix solving the nearest orthogonal approximation problem for a certain matrix, also maximizes the sum of the inner products between respective columns of the two aforementioned matrices. Thus, the vector $\boldsymbol{r}$ representing the rotation matrix $\boldsymbol{R} = \operatorname{mat}(\boldsymbol{r})$ that is closest to matrix $\boldsymbol{E} = \operatorname{mat}(\boldsymbol{e})$ will also maximize the inner product $\boldsymbol{r}^T\boldsymbol{e}$.

If $\boldsymbol{E} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ is a singular value decomposition (SVD), the orthogonal matrix nearest to $\boldsymbol{E}$ will be [5]

$$\boldsymbol{R} = \boldsymbol{U}\boldsymbol{C}\boldsymbol{V}^T,$$

where $\boldsymbol{C} = \operatorname{diag}(1, 1, \det(\boldsymbol{U}\boldsymbol{V}^T))$. Thus, to maximize the angle or, equivalently, the distance between $\boldsymbol{r}$ (corresponding to $\boldsymbol{R} = \operatorname{mat}(\boldsymbol{r})$) and $\boldsymbol{e}$, the following difference should be maximized:

$$\left\|\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T - \boldsymbol{U}\boldsymbol{C}\boldsymbol{V}^T\right\|_{\mathrm{F}} = \left\|\boldsymbol{U}(\boldsymbol{\Sigma} - \boldsymbol{C})\boldsymbol{V}^T\right\|_{\mathrm{F}},$$

where $||.||_{\mathrm{F}}$ denotes the Frobenius norm for matrices. Recalling that the squared Frobenius norm of a matrix equals the sum of its squared singular values, we observe that the squared Frobenius norm of the above difference is the squared sum of the diagonal entries of $\boldsymbol{\Sigma} - \boldsymbol{C}$. But we already know that the squared sum of the diagonal elements of $\boldsymbol{\Sigma}$ will be 3, because $\|\boldsymbol{e}\| = \sqrt{3}$ and, therefore, the squared sum of the singular values of $\boldsymbol{E}$ will be 3. Thus, the maximization problem can be stated as

$$\underset{\boldsymbol{\sigma}\in\mathbb{R}^3,\ \|\boldsymbol{\sigma}\|=\sqrt{3}}{\text{maximize}} \|\boldsymbol{\sigma}-\boldsymbol{c}\|^2,$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{c}$ are the vectors of the diagonal elements of $\boldsymbol{\Sigma}$ and $\boldsymbol{C}$. We observe that both vectors $\boldsymbol{\sigma}$ and $\boldsymbol{c}$ lie on the 3D sphere of radius $\sqrt{3}$. It follows that the distance between $\boldsymbol{\sigma}$ and $\boldsymbol{c}$ will be maximized when these points are antipodal, i.e. $\boldsymbol{\sigma} = -\boldsymbol{c}$. And since, due to symmetry, the maximum distance will always be the the same for any choice of $\boldsymbol{U}$ and $\boldsymbol{V}$, we choose $\boldsymbol{E} = -\boldsymbol{I}_3$. Subsequently, the rotation matrix closest to $\boldsymbol{E}$ will be

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Finally, the angle between $\boldsymbol{r}$ and $\boldsymbol{e}$ is computed as

$$\arccos(\boldsymbol{r}^T\boldsymbol{e}) = \arccos(1/3) \approx 70.529^\circ < 71^\circ,$$

which concludes the proof. □

5 Existence of exactly 4 inflection points in the direction of the projection of the gradient's component of descent on $\mathcal{O}(3)$ in the 90° region of $\boldsymbol{e}$

Proposition 5. *For $\boldsymbol{e} \in \mathbb{R}^9$ with $\|\boldsymbol{e}\|^2 = 1$, there exist exactly 4 vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \boldsymbol{\xi}_4$ with $\mathrm{mat}(\boldsymbol{\xi}_i) \in \mathcal{O}(3)$ in the 90° region of $\sqrt{3}\boldsymbol{e}$ for which the vectors $\sqrt{3}\boldsymbol{e} - \boldsymbol{\xi}_i$ are orthogonal to the tangent space of $\mathcal{O}(3)$ at $\boldsymbol{\xi}_i$.*

Proof. Let $\boldsymbol{r}$ be a vector such that $\mathrm{mat}(\boldsymbol{r}) \in \mathcal{SO}(3)$. We observe that

$$\left(\sqrt{3}\boldsymbol{e} - \boldsymbol{r}\right)^T \boldsymbol{N}_{\boldsymbol{r}} = 0 \iff \boldsymbol{e}^T\boldsymbol{N}_{\boldsymbol{r}} = 0, \tag{10}$$

where $\boldsymbol{N}_{\boldsymbol{r}} \in \mathbb{R}^{9\times 3}$ has the basis vectors of the tangent space of $\mathcal{SO}(3)$ at $\boldsymbol{r}$ in its columns. Clearly, since $-\boldsymbol{r} \in \mathcal{O}(3)$, it follows that the same holds for orthogonal matrices in general.

We are therefore looking for the set of orthogonal matrices that, in their vector form, are orthogonal to $\boldsymbol{e}$. This is equivalent to finding the minimizers of the nearest orthogonal matrix approximation problem [1]. Towards this end, of particular interest is Horn's solution [2] on the set of rotation matrices using quaternions. The minimizers are 4 eigenvectors of a symmetric data matrix that solve the first order conditions of the cost function. Being the eigenvectors of a symmetric matrix, these minimizing quaternions are orthogonal to each other. But this means that the relative rotation between them in pairs will be 180°. To show this, we denote two quaternions with $q_1 = (\rho_1,\ \boldsymbol{v}_1)$ and $q_2 = (\rho_2,\ \boldsymbol{v}_2)$ where $\rho_1, \rho_2 \in \mathbb{R}$ are the scalar parts and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^3$ the vector parts, respectively. Since the angle between q_1 and q_2 is 90°, it follows that the inner product $q_1 \cdot q_2$ of the two quaternions (as 4-vectors) will vanish. Thus,

$$q_1 \cdot q_2 = \rho_1\,\rho_2 + \boldsymbol{v}_1^T\,\boldsymbol{v}_2 = 0 \tag{11}$$

Denoting quaternion multiplication with $\odot$, the product $q_1^{-1} \odot q_2$ represents the relative rotation between q_1 and q_2. We observe that the scalar part of $q_1^{-1} \odot q_2$ is equal to $q_1 \cdot q_2$ and is therefore zero (cf. eq. (11)). Thus, if θ is the angle of the rotation, then $\cos(\theta/2) = 0$, which suggests that $\theta = 180°$.

We now denote the 4 solutions of the absolute orientation problem in $\mathcal{SO}(3)$ with $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, $\boldsymbol{r}_4$ and the respective matrices by $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{R}_3$, $\boldsymbol{R}_4$ in ascending order of eigenvalues (hence, distance from $\boldsymbol{e}$). A 180° relative rotation between the rotation matrices $\boldsymbol{R}_i$ and $\boldsymbol{R}_j$ is an operation where $\boldsymbol{R}_j$ is obtained by negating two of the rows in $\boldsymbol{R}_i$ and leaving the third unchanged. Thus, the inner product between $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ will be:

$$\boldsymbol{r}_i^T \boldsymbol{r}_j = 1 + (-1) + (-1) = -1, \tag{12}$$

which suggests that the angle between $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ is $\arccos\left(\boldsymbol{r}_i^T \boldsymbol{r}_j / 3\right) = \arccos(-1/3) \approx 109.471°$.

We know from Proposition 4 that $\boldsymbol{r}_1$ cannot lie further than 71° from $\boldsymbol{e}$ and therefore we may choose $\boldsymbol{\xi}_1 = \boldsymbol{r}_1$. However, given that the remaining 3 rotations lie 109.471° from each other, it follows they can be situated outside the 90° of $\boldsymbol{e}$. We therefore choose the remaining $\boldsymbol{\xi}_i$ as follows:

$$\boldsymbol{\xi}_i = \begin{cases} \boldsymbol{r}_i & \arccos(\boldsymbol{e}^T \boldsymbol{r}_i) \leq 90° \\ -\boldsymbol{r}_i & \text{otherwise} \end{cases}$$

Vectors $\boldsymbol{\xi}_i \in \mathcal{O}(3)$ are exactly 4 since they are derived from the 4 minimizers of the absolute orientation problem. □

6 Recovering the regional minimum by descending from $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, $\boldsymbol{\xi}_3$ and $\boldsymbol{\xi}_4$

Proposition 6. *For a minimizing eigenvector* $\boldsymbol{e}$ *of* $\boldsymbol{\Omega}$*, the feasible minimum in* $\mathcal{O}(3)$ *inside the* 90^o *region of* $\sqrt{3}\boldsymbol{e}$ *can be reached by descending from at least one of the vectors* $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \boldsymbol{\xi}_4$*, mentioned in Proposition 5.*

Proof. We know from Proposition 5 that the projection of the component of the gradient responsible for descent towards $\sqrt{3}\boldsymbol{e}$ changes its direction on the tangent space of $\mathcal{O}(3)$ exactly four times at $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \boldsymbol{\xi}_4$.

We deduce that any feasible path in the region between any two of $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \boldsymbol{\xi}_4$ will either contain a single local optimum or one local minimum and one local maximum, since any other configuration of local optima would imply the existence of an inflection in the projection of the component of descent of the gradient on the tangent space of $\mathcal{O}(3)$, which is a contradiction, as this point would have to be one of the remaining $\boldsymbol{\xi}_i$.

Similarly, we would anticipate that any feasible path in the region between two minima must be separated by an inflection in the projection of the gradient's component of descent on the tangent space of $\mathcal{O}(3)$ which is signified by one of $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \boldsymbol{\xi}_4$.

From the above, we conclude that descending on the feasible path for each $\boldsymbol{\xi}_i$ will exhaustively reveal the local minima in the region.

□

7 Unique solution of SQP in every iteration

Proposition 7. *Let $\boldsymbol{r} \in \mathbb{R}^9$ be the estimate of the rotation matrix which many not be feasible at some SQP iteration. If* $\operatorname{rank}(\boldsymbol{\Omega}) \geq 3$*, then the linearly constrained quadratic program*

$$\underset{\boldsymbol{\delta} \in \mathbb{R}^9}{\text{minimize}}\ \boldsymbol{\delta}^T \boldsymbol{\Omega} \boldsymbol{\delta} + 2\boldsymbol{r}^T \boldsymbol{\Omega} \boldsymbol{\delta} \ \text{ s.t. } \ \boldsymbol{H_r} \boldsymbol{\delta} = -\boldsymbol{h}(\boldsymbol{r}) \tag{13}$$

has a unique solution in $\mathbb{R}^9$.

Proof.

Part 1: Solution of the linearly constrained quadratic program when $\operatorname{rank}(\operatorname{mat}(\boldsymbol{r})) = 3$

Suppose $\boldsymbol{r}$ is the current estimate of the unknown rotation at some step of the SQP process, such that $\operatorname{rank}(\operatorname{mat}(\boldsymbol{r})) = 3$. It follows from proposition 2 that $\operatorname{rank}(\boldsymbol{H_r}) = 6$. Let $\boldsymbol{U} \in \mathbb{R}^{9\times 6}$ be the matrix containing a basis of the row space of $\boldsymbol{H_r}$ as columns and similarly, let $\boldsymbol{N} \in \mathbb{R}^{9\times 3}$ be a matrix whose columns are a set of basis vectors of the null space of $\boldsymbol{H_r}$. To solve the linearly constrained quadratic, we parametrize $\boldsymbol{\delta}$ using one component $\boldsymbol{\delta}_N$ in the null space and one component $\boldsymbol{\delta}_H$ in the row space of $\boldsymbol{H_r}$, as follows:

$$\boldsymbol{\delta} = \boldsymbol{\delta}_N + \boldsymbol{\delta}_H = \boldsymbol{N}\boldsymbol{\alpha} + \boldsymbol{U}\boldsymbol{\beta}, \tag{14}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^3$ and $\boldsymbol{\beta} \in \mathbb{R}^6$.

Substituting eq. (14) into the linear constraint of eq. (13) yields

$$\boldsymbol{H_r N}\boldsymbol{\alpha} + \boldsymbol{H_r U}\boldsymbol{\beta} = -\boldsymbol{h}(\boldsymbol{r}). \tag{15}$$

Since $\mathbf{N}$ contains the basis vectors of $\operatorname{null}(\boldsymbol{H_r})$, it follows that the term $\boldsymbol{H_r N}\boldsymbol{\alpha}$ in eq. (14) vanishes and we can solve for $\boldsymbol{\beta}$

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{H_r U})^{-1} \boldsymbol{h}(\boldsymbol{r}). \tag{16}$$

We may now substitute $\boldsymbol{\delta}$ in the quadratic cost function of the linearly constrained problem of (13) and obtain the first order optimality conditions in terms of $\boldsymbol{\alpha}$:

$$\boldsymbol{N}^T \boldsymbol{\Omega} \boldsymbol{N}\boldsymbol{\alpha} = \boldsymbol{N}^T \boldsymbol{\Omega} \left(\boldsymbol{r} - \boldsymbol{U}(\boldsymbol{H_r U})^{-1} \boldsymbol{h}(\boldsymbol{r})\right) \tag{17}$$

At this point, eq. (17) will have a unique solution if $\boldsymbol{N}$ is disjoint with the null space of $\boldsymbol{\Omega}$. We can prove that this will always be the case if the SQP iteration is initiated from a point that is not an eigenvector of $\boldsymbol{\Omega}$. Indeed, it is straightforward to infer that the null space of the proper orthonormality constraints is also a subset of the tangent space of $\boldsymbol{r}$ because $\boldsymbol{r}$ can be written as a linear combination of the rows of $\boldsymbol{H_r}$[1]. This also means that the columns of $\boldsymbol{N}$ are in the tangent space of $\boldsymbol{r}/\|\boldsymbol{r}\|$ on the unit 8-sphere. But if the tangent space contains a null space vector of $\boldsymbol{\Omega}$, it follows that $\boldsymbol{r}/\|\boldsymbol{r}\|$ must be orthogonal to it, which implies that $\boldsymbol{r}/\|\boldsymbol{r}\|$ should itself be the eigenvector of $\boldsymbol{\Omega}$ from which we obtained the initial feasible solution as the nearest rotation matrix (which lies less than $90°$ away from any other eigenvector), which is a contradiction. Thus, the tangent space of $\boldsymbol{r}$ is disjoint with the null space of $\boldsymbol{\Omega}$.

[1] Specifically, the first 3 rows.

Before solving for $\boldsymbol{\alpha}$ we need to ensure that $\boldsymbol{N}^T\boldsymbol{\Omega}\boldsymbol{N}$ is a full-rank 3×3 matrix. Without harm of generality, suppose that $\operatorname{rank}(\boldsymbol{\Omega}) = 3$. Then, since $\boldsymbol{\Omega}$ is a positive semi-definite matrix, it can be written as follows:

$$\boldsymbol{\Omega} = s_1 \boldsymbol{e}_1 \boldsymbol{e}_1^T + s_2 \boldsymbol{e}_2 \boldsymbol{e}_2^T + s_3 \boldsymbol{e}_3 \boldsymbol{e}_3^T,$$

where $s_1 \geq s_2 \geq s_3 > 0$ are its non-vanishing eigenvalues and $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ the respective eigenvectors. For $\boldsymbol{N}^T\boldsymbol{\Omega}\boldsymbol{N}$ to be invertible, the vectors $\boldsymbol{N}^T\boldsymbol{e}_1$, $\boldsymbol{N}^T\boldsymbol{e}_2$ and $\boldsymbol{N}^T\boldsymbol{e}_3$ should be linearly independent. If they were not independent, then there would exist scalars $\kappa, \lambda, \mu \in \mathbb{R}$ with $|\kappa| + |\lambda| + |\mu| \neq 0$, such that $\boldsymbol{N}^T \left(\kappa \boldsymbol{e}_1 + \lambda \boldsymbol{e}_2 + \mu \boldsymbol{e}_3\right) = \boldsymbol{0}$. The latter implies that there would exist a vector in the hyperplane spanned by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ that would be orthogonal to all the basis vectors in $\boldsymbol{N}$. But the only way this could be true, would be if the columns of $\boldsymbol{N}$ belonged to the null space of $\boldsymbol{\Omega}$, which is again a contradiction.

Having established from the above that $\boldsymbol{N}^T\boldsymbol{\Omega}\,\boldsymbol{N}$ is invertible, the solution for $\boldsymbol{\alpha}$ is

$$\hat{\boldsymbol{\alpha}} = \left(\boldsymbol{N}^T\boldsymbol{\Omega}\boldsymbol{N}\right)^{-1} \boldsymbol{N}^T\boldsymbol{\Omega} \left(\boldsymbol{r} - \boldsymbol{U}\left(\boldsymbol{G}_{\boldsymbol{r}}\boldsymbol{U}\right)^{-1} \boldsymbol{g}(\boldsymbol{r})\right). \tag{18}$$

Part 2: The rotation estimate always represents a full-rank matrix

Thus far, we have shown that if the SQP is initialized from the nearest feasible point to an eigenvector of $\boldsymbol{\Omega}$ and the descent leads to a full-rank approximate rotation, then the linearly constrained quadratic program at the current step has a unique solution given by eqs. (14), (16) and (18). We now have to show that the new estimate will always be a full-rank matrix, provided that the previous one was.

First, we need to show that if $\operatorname{rank}(\boldsymbol{r}) \geq 2$, then any non-trivial perturbation (i.e., not zero) of the matrix on the null space of the Jacobian of the proper orthonormality function will be a full-rank matrix. In other words, the following

$$\operatorname{rank}\left(\operatorname{mat}\left(\boldsymbol{r} + \boldsymbol{N}\,\boldsymbol{\alpha}\right)\right) = 3$$

should hold for any vector $\boldsymbol{\alpha} \in \mathbb{R}^3 - \{\boldsymbol{0}\}$. To prove this claim, we compute the null space of $\boldsymbol{H}_{\boldsymbol{r}}$ assuming, without loss of generality, that the minor determinant of $\operatorname{mat}(\boldsymbol{r})$ corresponding to element r_{13} is not zero. We now distinguish the following cases regarding element r_{31}:

For $r_{31} \neq 0$, the null space of $\boldsymbol{H}_{\boldsymbol{r}}$ is

$$\boldsymbol{N} = \operatorname{null}\left(\boldsymbol{H}_{\boldsymbol{r}}\right) = \begin{bmatrix} \frac{r_{12}r_{33} - r_{13}r_{32}}{d_{13}} & -\frac{r_{12}}{r31} & -\frac{r_{12}r21r33 - r_{13}r_{22}r_{31}}{r_{31}d_{13}} \\ -\frac{r_{11}r_{33} - r_{13}r_{31}}{d_{13}} & \frac{r_{11}}{r_{31}} & \frac{r_{21}(r11r_{33} - r_{13}r_{31})}{r_{31}d_{13}} \\ \frac{r_{11}r_{32} - r_{12}r_{31}}{d_{13}} & 0 & -\frac{r_{11}r_{22} - r_{12}r_{21}}{d_{13}} \\ \frac{r_{22}r_{33} - r_{23}r_{32}}{d_{13}} & -\frac{r_{22}}{r_{31}} & -\frac{r_{22}(r_{21}r_{33} - r_{23}r_{31})}{r_{31}d_{13}} \\ -\frac{r_{21}r_{33} - r_{23}r_{31}}{d_{13}} & \frac{r21}{r31} & \frac{r_{21}(r_{21}r_{33} - r_{23}r_{31})}{r_{31}d_{13}} \\ 1 & 0 & 0 \\ 0 & -\frac{r_{32}}{r_{31}} & -\frac{r_{33}}{r_{31}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

where d_{13} is the minor determinant of $\operatorname{mat}(\boldsymbol{r})$ at the indexed element:

$$d_{13} = \det \begin{bmatrix} r_{21} & r_{22} \\ r_{31} & r_{32} \end{bmatrix}$$

We claim that the equation

$$\det\left(\operatorname{mat}\left(\boldsymbol{N}\,\boldsymbol{\alpha}+\boldsymbol{r}\right)\right)=0$$

has no solution for $\boldsymbol{\alpha}$ in $\mathbb{R}^3-\{\boldsymbol{0}\}$. Indeed, substituting eq. (19) into eq. (7) yields the following solutions for the first component of $\boldsymbol{\alpha}$

$$\alpha_1=\pm\frac{\left(r_{21}r_{32}-r_{22}r_{31}\right)\sqrt{-r_{31}^2\left(r_{31}^2+r_{32}^2+r_{33}^2\right)}}{r_{31}\left(r_{31}^2+r_{32}^2+r_{33}^2\right)}, \tag{20}$$

which are not real numbers.

The case $r_{31}=0$ can be dealt with in a similar manner. Since we have assumed that $d_{13}\neq 0$, it follows that $r_{21}\neq 0$ and $r_{23}\neq 0$. We can therefore swap the first row with the third and obtain an equivalent problem in which the first element of the third row will be non-zero so that the results of eq. (20) apply to this case as well.

We have thus established that any non-trivial perturbation $\boldsymbol{\delta}_N$ on the null space of $\boldsymbol{H}_{\boldsymbol{r}}$ leads to a new vector $\boldsymbol{r}+\boldsymbol{\delta}_N$ which is a representation of a full-rank 3×3 matrix.

We examine next the second component of motion $\boldsymbol{\delta}_H=\boldsymbol{U}\,\boldsymbol{\beta}$ in eq. (14), lying in the row space of $\boldsymbol{H}_{\boldsymbol{r}}$. This motion is determined by the value of the proper orthonormality function $\boldsymbol{h}(\boldsymbol{r})$, according to eqs. (14), (15) and (16). We now claim that if $\operatorname{rank}(\operatorname{mat}(\boldsymbol{r}))=3$ then $\operatorname{rank}(\operatorname{mat}(\boldsymbol{r}+\boldsymbol{\delta}_H))\geq 2$. To prove it, we assume that the second and third rows of $\operatorname{mat}\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)$ are multiples of the first:

$$\begin{aligned}\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{4:6}&=\kappa\,\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{1:3}\\ \left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{7:9}&=\lambda\,\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{1:3}\end{aligned} \tag{21}$$

We also know from eq.(15) that the following should hold:

$$\begin{aligned}\boldsymbol{r}_{1:3}^T\boldsymbol{\delta}_{H\,1:3}=1-\boldsymbol{r}_{1:3}^T\boldsymbol{r}_{1:3}\quad&\Leftrightarrow\quad\boldsymbol{r}_{1:3}^T\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{1:3}=1\\ \boldsymbol{r}_{4:6}^T\boldsymbol{\delta}_{H\,4:6}=1-\boldsymbol{r}_{4:6}^T\boldsymbol{r}_{4:6}\quad&\Leftrightarrow\quad\boldsymbol{r}_{4:6}^T\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{4:6}=1\\ \boldsymbol{r}_{1:3}^T\boldsymbol{\delta}_{H\,4:6}+\boldsymbol{r}_{4:6}^T\boldsymbol{\delta}_{H\,1:3}=-\boldsymbol{r}_{1:3}^T\boldsymbol{r}_{4:6}\quad&\Leftrightarrow\quad\boldsymbol{r}_{1:3}^T\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{4:6}+\boldsymbol{r}_{4:6}^T\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{1:3}=0\\ \boldsymbol{r}_{1:3}^T\boldsymbol{\delta}_{H\,7:9}+\boldsymbol{r}_{7:9}^T\boldsymbol{\delta}_{H\,1:3}=-\boldsymbol{r}_{1:3}^T\boldsymbol{r}_{7:9}\quad&\Leftrightarrow\quad\boldsymbol{r}_{1:3}^T\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{7:9}+\boldsymbol{r}_{7:9}^T\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{1:3}=0\\ \boldsymbol{r}_{4:6}^T\boldsymbol{\delta}_{H\,7:9}+\boldsymbol{r}_{7:9}^T\boldsymbol{\delta}_{H\,4:6}=-\boldsymbol{r}_{4:6}^T\boldsymbol{r}_{7:9}\quad&\Leftrightarrow\quad\boldsymbol{r}_{4:6}^T\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{7:9}+\boldsymbol{r}_{7:9}^T\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{4:6}=0\end{aligned} \tag{22}$$

It is straightforward to show that a contradiction arises from eqs. (21) and (22). First, we observe that $\kappa\neq 0$ so that the two linearized norm constraints (i.e., the first two equations in (22)) may hold. Thus, for $\kappa\neq 0$ we substitute into the third equation in (22) (i.e., the linearized orthogonality constraint between the first and the second row of $\operatorname{mat}(\boldsymbol{r}+\boldsymbol{\delta})$) and obtain:

$$\left(\kappa\,\boldsymbol{r}_{1:3}+\boldsymbol{r}_{4:6}\right)^T\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{1:3}=0 \tag{23}$$

Eq. (23) clearly states that the first row $\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{1:3}$ of the perturbed matrix is orthogonal to the plane spanned by $\boldsymbol{r}_{1:3}$ and $\boldsymbol{r}_{4:6}$. In a similar manner, we obtain

$$\left(\lambda\,\boldsymbol{r}_{1:3}+\boldsymbol{r}_{7:9}\right)^T\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{1:3}=0, \tag{24}$$

which implies that the first row $\left(\boldsymbol{r}+\boldsymbol{\delta}_H\right)_{1:3}$ of the perturbed matrix is also orthogonal to either the plane spanned by $\boldsymbol{r}_{1:3}$ and $\boldsymbol{r}_{7:9}$ (when $\lambda\neq 0$), or the vector $\boldsymbol{r}_{7:9}$ (when $\lambda=0$). In either case, this is a contradiction because by assumption, $\operatorname{rank}\left(\operatorname{mat}(\boldsymbol{r})\right)=3$ and therefore, in the first case $(\lambda=0)$

the first row of the perturbed matrix ends up being normal to two disjoint 3D planes, while in the second case ($\lambda \neq 0$) the first row of the perturbed matrix is orthogonal to all rows of a rank-3 matrix.

We have thus concluded that starting an SQP descent from a full-rank estimate that is not an eigenvector of $\mathbf{\Omega}$ but is less than 90° away from it, will subsequently lead to a sequence of refined estimates that will also be full-rank and at the same time, confined in a region of no more than 90° away from the eigenvector used to determine the initial solution. In other words, the SQP linear system will always have a unique solution, provided that the starting point is the nearest rotation matrix to an eigenvector of $\mathbf{\Omega}$. □

8 Additional comparative plots

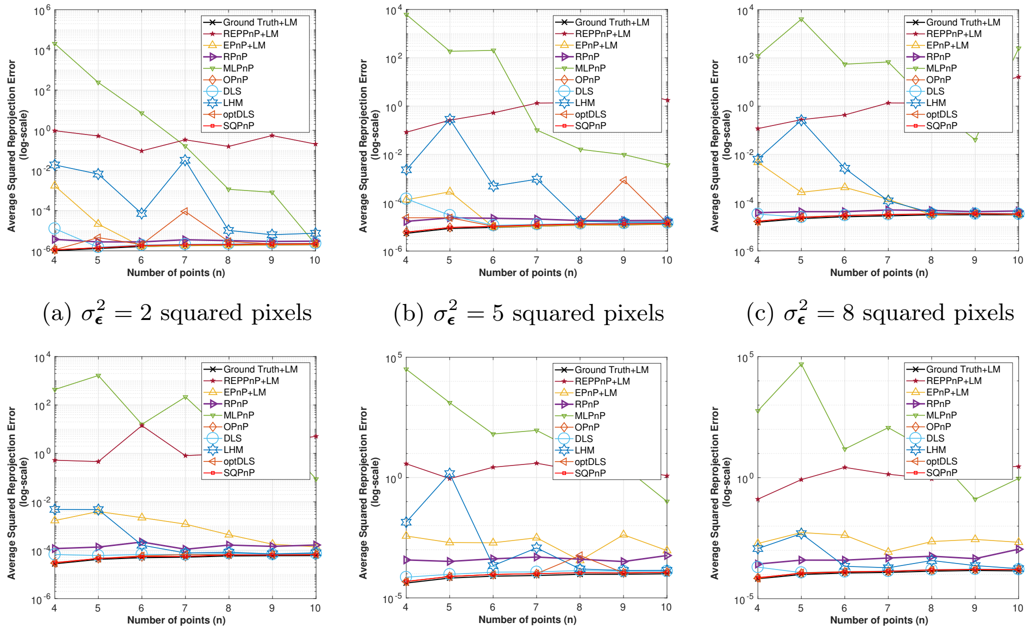
To provide further insight in the performance of SQPnP relative to the tested PnP solvers, this section presents additional plots derived from exactly the same synthetic experiments summarized in Figure 2 of the main manuscript. More details are in the following subsections.

8.1 Plots of average squared reprojection error

Figure 1 in the present document consists of plots illustrating the average squared reprojection error for the experiments reported in the main manuscript. Using the average instead of the maximum as metric, results in these plots having a smoother profile compared to the plots corresponding to the maximum squared error (i.e., those in Fig. 2 in the main manuscript). Much like in the maximum error plots, methods that are more susceptible to noise clearly stand out in the average error plots from those that are more robust. Nevertheless, the maximum plots are far more informative regarding the consistency and the accuracy achieved by the more robust methods. In particular, we observe from the maximum error plots (Fig. 2 in the main manuscript) that RPnP and DLS are generally not as consistently accurate as SQPnP and OPnP, whereas in the average error plots (Fig. 1 in this document), these methods exhibit convergence profiles which are practically indistinguishable from each other.

8.2 Plots of translation and rotation error

Figures 2 and 3 in this document illustrate respectively the maximum translation and rotation errors for all experiments and solvers in terms of the number of points n. For completeness, the average translation and rotation errors are also shown in Figures 4 and 5, respectively. Similarly to the case of maximum and average reprojection error plots, there is no significant discrepancy in the overall performance profile of each PnP method between maximum and average error plots. However, it is worth noting that maximum error plots tend to provide a clearer picture regarding the consistency of convergence of the more accurate methods. Note that occasional spikes in the maximum divergence in the translation and/or rotation error plots for $n = 4$ despite that the corresponding squared error is very low, is a consequence of the fact that increased levels of noise actually skew the minimum away from the ground truth.



(a) $\sigma_{\epsilon}^2 = 2$ squared pixels (b) $\sigma_{\epsilon}^2 = 5$ squared pixels (c) $\sigma_{\epsilon}^2 = 8$ squared pixels

(d) $\sigma_{\epsilon}^2 = 11$ squared pixels (e) $\sigma_{\epsilon}^2 = 14$ squared pixels (f) $\sigma_{\epsilon}^2 = 17$ squared pixels

Figure 1: Plots of average squared reprojection error for 500 executions of each PnP solver on n random points, $4 \leq n \leq 10$. For each n, the points are repeatedly sampled from a previously generated point population contaminated with additive Gaussian noise. Each plot represents the results obtained by points drawn from a different population and whose projections were contaminated with zero-mean Gaussian noise of variance $\sigma_{\epsilon}^2 \in \{2, 5, 8, 11, 14, 17\}$ squared pixels.

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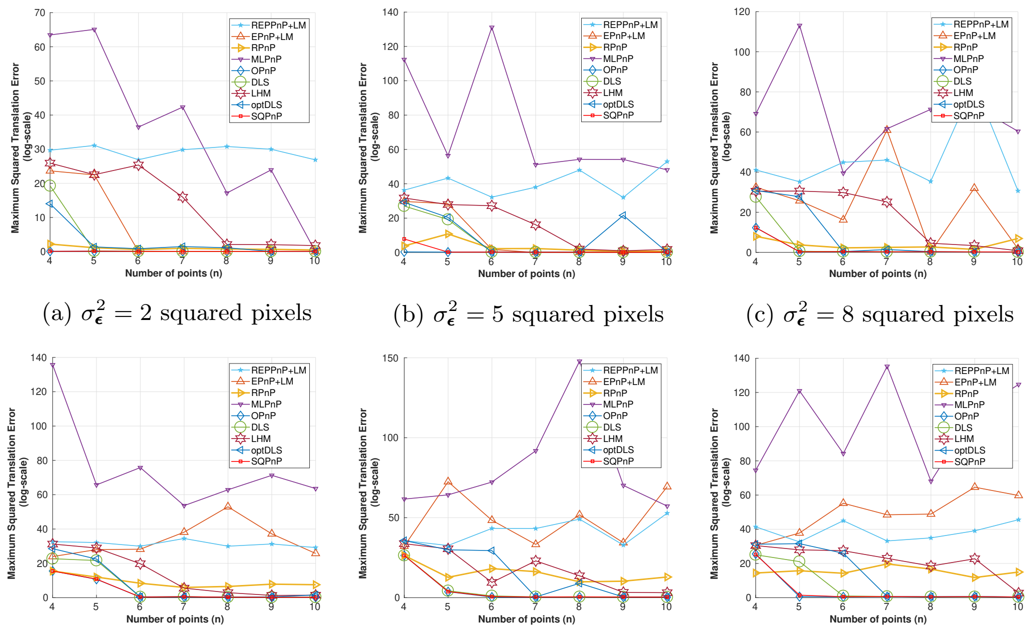
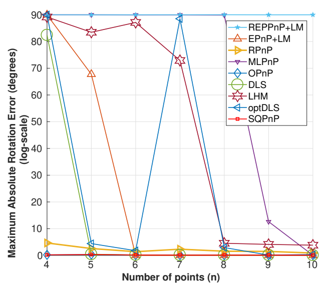


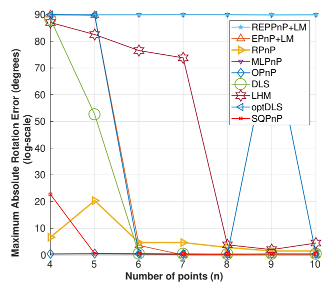
(a) $\sigma_{\boldsymbol{\epsilon}}^2 = 2$ squared pixels (b) $\sigma_{\boldsymbol{\epsilon}}^2 = 5$ squared pixels (c) $\sigma_{\boldsymbol{\epsilon}}^2 = 8$ squared pixels

(d) $\sigma_{\boldsymbol{\epsilon}}^2 = 11$ squared pixels (e) $\sigma_{\boldsymbol{\epsilon}}^2 = 14$ squared pixels (f) $\sigma_{\boldsymbol{\epsilon}}^2 = 17$ squared pixels

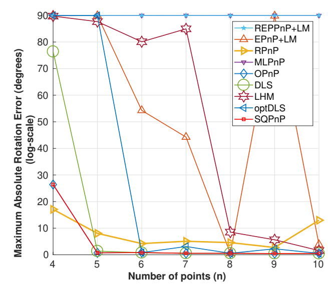
Figure 2: Plots of maximum translation error (in meters) over 500 executions of each PnP solver on n randomly sampled points, $4 \leq n \leq 10$, contaminated with zero-mean Gaussian noise of variance $\sigma_{\boldsymbol{\epsilon}}^2 \in \{2, 5, 8, 11, 14, 17\}$ squared pixels.



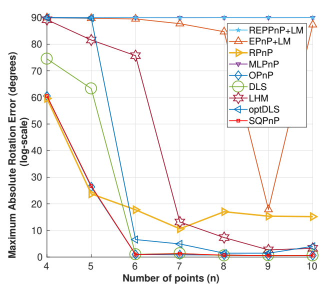
(a) $\sigma_{\boldsymbol{\epsilon}}^2 = 2$ squared pixels



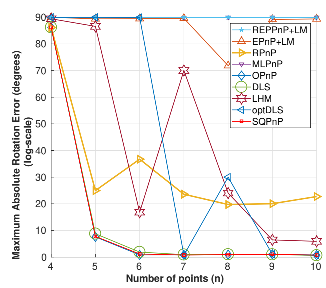
(b) $\sigma_{\boldsymbol{\epsilon}}^2 = 5$ squared pixels



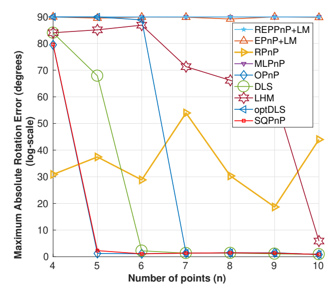
(c) $\sigma_{\boldsymbol{\epsilon}}^2 = 8$ squared pixels



(d) $\sigma_{\boldsymbol{\epsilon}}^2 = 11$ squared pixels

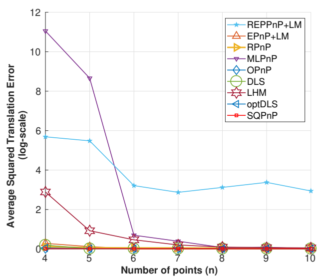


(e) $\sigma_{\boldsymbol{\epsilon}}^2 = 14$ squared pixels

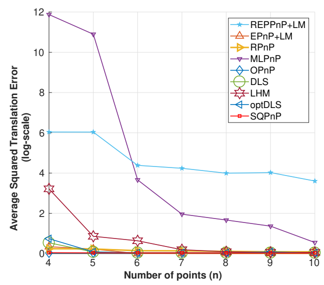


(f) $\sigma_{\boldsymbol{\epsilon}}^2 = 17$ squared pixels

Figure 3: Plots of maximum rotation error (in degrees, computed as the absolute angle between quaternions) for 500 executions of each PnP solver on n randomly sampled points, $4 \leq n \leq 10$, contaminated with zero-mean Gaussian noise of variance $\sigma_{\boldsymbol{\epsilon}}^2 \in \{2, 5, 8, 11, 14, 17\}$ squared pixels.



(a) $\sigma_{\boldsymbol{\epsilon}}^2 = 2$ squared pixels



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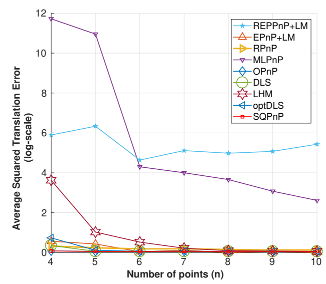
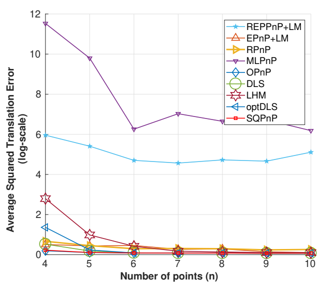
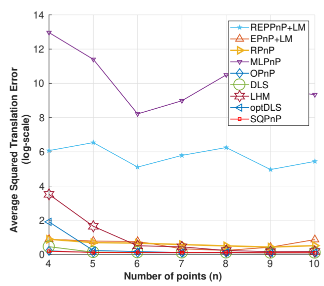


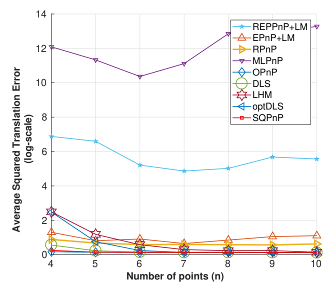
(c) $\sigma_{\boldsymbol{\epsilon}}^2 = 8$ squared pixels



(d) $\sigma_{\boldsymbol{\epsilon}}^2 = 11$ squared pixels

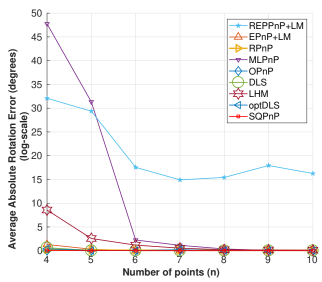


(e) $\sigma_{\boldsymbol{\epsilon}}^2 = 14$ squared pixels

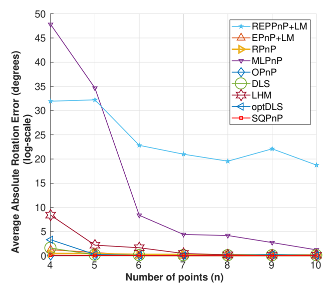


(f) $\sigma_{\boldsymbol{\epsilon}}^2 = 17$ squared pixels

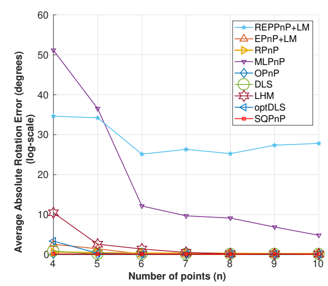
Figure 4: Plots of average translation error (in meters) for 500 executions of each PnP solver on n randomly sampled points, $4 \leq n \leq 10$, contaminated with zero-mean Gaussian noise of variance $\sigma_{\boldsymbol{\epsilon}}^2 \in \{2, 5, 8, 11, 14, 17\}$ squared pixels.



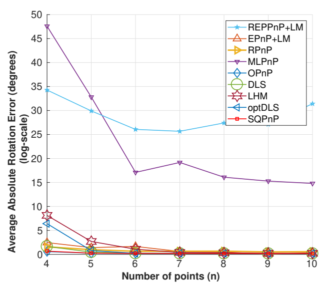
(a) $\sigma_{\boldsymbol{\epsilon}}^2 = 2$ squared pixels



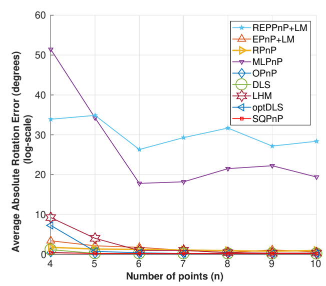
(b) $\sigma_{\boldsymbol{\epsilon}}^2 = 5$ squared pixels



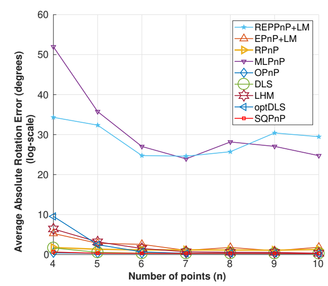
(c) $\sigma_{\boldsymbol{\epsilon}}^2 = 8$ squared pixels



(d) $\sigma_{\boldsymbol{\epsilon}}^2 = 11$ squared pixels



(e) $\sigma_{\boldsymbol{\epsilon}}^2 = 14$ squared pixels



(f) $\sigma_{\boldsymbol{\epsilon}}^2 = 17$ squared pixels

Figure 5: Plots of average rotation error (in degrees, computed as the absolute angle between quaternions) for 500 executions of each PnP solver on n randomly sampled points, $4 \leq n \leq 10$ contaminated with zero-mean Gaussian noise of variance $\sigma_{\boldsymbol{\epsilon}}^2 \in \{2, 5, 8, 11, 14, 17\}$ squared pixels.