A Path Constraints

Lemma 2 (Paths Constraints). Let $G' = (V', E', w) = (V \cup T, E \cup E^S, w)$ be an edge-weighted graph extended by terminal nodes $T$. For any edge indicator $y \in 0, 1^{\left|E \cup S\right|}$ that satisfies

$$\sum_{t \in T} y_{tv} = \left|T\right| - 1 \quad \forall v \in V$$

the following set of constraints are equivalent:

$$y_{ut} + y_{uv} + y_{vt'} \geq 1 \quad \forall (u, v) \in E \forall t, t' \in T, t \neq t' \quad (37)$$

$$y_{tu} + y_{uv} \geq y_{tv}, \quad \forall uv \in E, t \in T \quad (38)$$

$$y_{tv} + y_{uv} \geq y_{tu}, \quad \forall uv \in E, t \in T. \quad (39)$$

Proof. This lemma is trivially fulfilled for $|T| \leq 1$. We will prove the lemma for $|T| > 1$ by contradiction in each direction.

"⇒" Assume eq. (37) holds and $\exists y_{tv} > y_{tu} + y_{uv}$. In case $y_{tv} = 1$, eq. (36) implies that $\exists t' \neq t : y_{tv} = 0$ which leads to the contradiction

$$y_{tv} > y_{tu} + y_{uv} \geq 1 - y_{tu} = 1$$

In case $y_{tv} = 0$, eq. (36) implies that $\forall t' \neq t : y_{tv} = 0$ leading to the contradiction

$$y_{tv} > y_{tu} + y_{uv} \geq 1 - y_{tu} = 0.$$  

The proof for eq. (39) is analogous.

"⇐" Assume eqs. (38) and (39) hold and $\exists (u, v) \in E t, t' \in T, t \neq t' : y_{ut} + y_{uv} + y_{vt'} < 1$. This leads to the contradiction

$$y_{ut} + y_{uv} + y_{vt'} < 1$$

$$\Rightarrow y_{ut} = 0, y_{uv} = 0 \quad \text{and} \quad y_{vt'} = 0 \Rightarrow y_{vt} = 1$$

$$\Rightarrow 1 = y_{vt} \geq y_{ut} + y_{uv} = 0.$$