Deep Positional and Relational Feature Learning for Rotation-Invariant Point Cloud Analysis (Supplemental Materials)

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1 Proof of rotation-invariance for geometric feature

Proof. Given points $c, p, q \in \mathcal{R}^3$ in Fig. 1 of our paper, for the distance in geometric features, e.g., $|\overrightarrow{cq}|$, after a random rotation R with $R^\top R = I$, we have

$$\begin{aligned} |\overrightarrow{(Rc)(Rq)}| &= \sqrt{(Rq - Rc)^\top (Rq - Rc)} \\ &= \sqrt{(q-c)^\top R^\top R (q-c)} \\ &= \sqrt{(q-c)^\top (q-c)} \\ &= |\overrightarrow{cq}|. \end{aligned} \tag{1}$$

For the cosine value of angle α, the new angle $\hat{\alpha}$ after rotation with rotation matrix R, we have

$$\begin{aligned} cos(\hat{\alpha}) &= \frac{\overrightarrow{(Rc)(Ro)}^\top \overrightarrow{(Rc)(Rq)}}{|\overrightarrow{(Rc)(Ro)}||\overrightarrow{(Rc)(Rq)}|} \\ &= \frac{\overrightarrow{co}^\top \overrightarrow{cq}}{|\overrightarrow{co}||\overrightarrow{cq}|} \\ &= cos(\alpha) \end{aligned} \tag{2}$$

The rotation-invariance of other distance or angle terms in geometric feature g (Sect. 3.1) can be similarly proved. Therefore, as a concatenation of rotation-invariant distances and angles, g is also rotation-invariant.

2 Proof of Theorem 1

Before the proof of Theorem 1, we first present following two lemmas.

Lemma 1. $G_S = \{g_j |\ g_j = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \alpha, \beta, \gamma \in \{-1, 1\}\}$ *constitutes a group with group operation "·" as matrix multiplication, i.e.,* $\forall g_i, g_j \in G_S, g_i \cdot g_j = g_i g_j$.

Proof. According to definition of group, G_S satisfies following four conditions.

- There exists identity element, i.e, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- For any element $g_i = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$, there exists its inverse element $g_i^{-1} = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix}$, and because $\alpha, \beta, \gamma \in \{-1, 1\}$, we have $g_i^{-1} = g_i \in G_S$.
- Closure. $\forall g_i, g_j \in G_S$, we have

$$\begin{aligned} g_i \cdot g_j &= g_i g_j \\ &= \begin{bmatrix} \alpha_i & 0 & 0 \\ 0 & \beta_i & 0 \\ 0 & 0 & \gamma_i \end{bmatrix} \begin{bmatrix} \alpha_j & 0 & 0 \\ 0 & \beta_j & 0 \\ 0 & 0 & \gamma_j \end{bmatrix} \\ &= \begin{bmatrix} \alpha_i\alpha_j & 0 & 0 \\ 0 & \beta_i\beta_j & 0 \\ 0 & 0 & \gamma_i\gamma_j \end{bmatrix} \end{aligned} \tag{3}$$

  Considering that $\alpha_i, \alpha_j \in \{1, -1\}$, we have $\alpha_i\alpha_j \in \{1, -1\}$, similarly, $\beta_i\beta_j, \gamma_i\gamma_j \in \{1, -1\}$. So $g_i \cdot g_j \in G_S$.
- Associativity. $\forall g_i, g_j, g_k \in G$, we need to prove $(g_i \cdot g_j) \cdot g_k = g_i \cdot (g_j \cdot g_k)$. Considering that "$\cdot$" is defined as matrix multiplication, it is obvious that $(g_i \cdot g_j) \cdot g_k = (g_i g_j) g_k = g_i(g_j g_k) = g_i \cdot (g_j \cdot g_k)$.

In summary, G_S satisfies the necessary conditions of a group, so it is a group.

Lemma 2. *For group* $G_S = \{g_j |\ g_j = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \alpha, \beta, \gamma \in \{-1, 1\}\}$ *with group operation as matrix multiplication, we have* $g_j G_S = G_S$, $\forall g_j \in G_S$.

Proof. We proof this lemma by following two steps.

Step 1. We first prove $g_j G_S \subset G_S$. For $\forall b \in G_S$, because $g_j \in G_S$ and G_S is a group, we have $g_j b \in G_S$, and $g_j G_S \subset G_S$.

Step 2. We then prove $G_S \subset g_j G_S$. Considering $g_j \in G_S$ and G_S is a group, the inverse element of $g_j \in G_S$ exists, which is denoted as g_j^{-1}. Then for $\forall b \in G_S$, we have $g_j^{-1} b \in G_S$. Then

$$b = g_j g_j^{-1} b = g_j (g_j^{-1} b) \tag{4}$$

Because $(g_j^{-1} b) \in G_S$, we have $g_j(g_j^{-1} b) \in g_j G_S$, i.e., $b \in g_j G_S$, $G_S \subset g_j G_S$.

By Step 1-2, it is obvious that $G_S = g_j G_S$.

With above two lemmas, we now prove Theorem 1.

Theorem 1. *Denoting $P \in \mathcal{R}^{N\times 3}$ as point cloud of a shape, its sym-space $H_S(P)$ and rot-space $H_{SR}(P)$ are rotation-invariant for P, i.e., for rotated shape $Q = PR$ of P with any rotation matrix $R \in \mathcal{R}^{3\times 3}$, we have $H_S(P) = H_S(Q)$ and $H_{SR}(P) = H_{SR}(Q)$. Assuming the pose selector is injective, the selected pose of P is rotation-invariant, i.e., $\widetilde{P} = \widetilde{Q}$.*

Proof. We conduct PCA-normalization on point cloud P with Singular Value Decomposition (SVD), the normalized point cloud is

$$\widehat{P} = PU^P, \text{ where } [U^P, \Lambda^P, (U^P)^\top] = \text{SVD}(P^\top P), \tag{5}$$

$U^P = [u_1^P, u_2^P, u_2^P], \Lambda^P = \text{diag}\{\lambda_1^P, \lambda_2^P, \lambda_3^P\}$ are respectively matrix of singular vectors and singular values. Similarly, for rotated point cloud Q, we have $\widehat{Q} = QU^Q$, where $[U^Q, \Lambda^Q, (U^Q)^\top] = \text{SVD}(Q^\top Q)$.

We next prove Theorem 1 by four steps.

Step 1. We first prove that λ_i^Q ($i \in \{1, 2, 3\}$) is the singular-value of $P^\top P$ with corresponding singular-vector as Ru_i^Q. Based on Eqn. (5), we have

$$u_i^P \lambda_i^P = P^\top P u_i^P, \quad u_i^Q \lambda_i^Q = Q^\top Q u_i^Q. \tag{6}$$

Considering that

$$Q^\top Q = R^\top P^\top P R, \tag{7}$$

we can derive that

$$u_i^Q \lambda_i^Q = Q^\top Q u_i^Q = R^\top P^\top P R u_i^Q. \tag{8}$$

Because R is a rotation matrix satisfying $RR^\top = I$, we have

$$R u_i^Q \lambda_i^Q = RR^\top P^\top P R u_i^Q = P^\top P R u_i^Q, \tag{9}$$

that is $(Ru_i^Q)\lambda_i^Q = (P^\top P)(Ru_i^Q)$. This tells us that λ_i^Q is the singular-value of $P^\top P$ with corresponding singular-vector as Ru_i^Q. We can similarly prove that λ_i^P is the singular-value of $Q^\top Q$ with corresponding singular-vector as $R^\top u_i^P$.

Step 2. We then prove that $H_S(P) = H_S(Q)$.

By step 1, for $i \in \{1, 2, 3\}$, $R^\top u_i^P$ is the singular-vector of $Q^\top Q$ with corresponding singular-value λ_i^Q. Since both $R^\top u_i^P$ and $-R^\top u_i^P$ can be the eigenvector of $Q^\top Q$ corresponding to eigen-value λ_i^Q, we have $u_i^Q \in \{R^\top u_i^P, -R^\top u_i^P\}$. Then we can derive that

$$Qu_i^Q \in \{QR^\top u_i^P, -QR^\top u_i^P\}, \tag{10}$$

that is

$$QU^Q \in \{QR^\top U^P g_j |\ g_j = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \alpha, \beta, \gamma \in \{-1, 1\}\}. \tag{11}$$

Considering that $Q = PR$, $QR^\top = PRR^\top = P$, we have

$$QU^Q \in \{PU^P g_j \mid g_j = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \alpha, \beta, \gamma \in \{-1, 1\}\}. \tag{12}$$

By Eqn.(12), there exists a g_j such that $QU^Q = PU^P g_j$, i.e., $\widehat{Q} = \widehat{P} g_j$. Assume that $G_S(\cdot)$ is group action on a point cloud in bracket defined as matrix multiplication between elements of group G_S and point cloud, then we can derive

$$\begin{aligned} G_S(\widehat{Q}) &= G_S(\widehat{P} g_j) \\ &= \{\widehat{P} g_j g_k \mid g_k \in G_S\} \\ &= (g_j G_S)(\widehat{P}) \\ &= G_S(\widehat{P}), \end{aligned} \tag{13}$$

where the fourth equality holds by Lemma 2. By definition of sym-space in Sect. 3.2, it is obvious that $G_S(\widehat{Q})$ and $G_S(\widehat{P})$ are just the sym-space $H_S(Q)$ and $H_S(P)$ respectively, and then we can derive that $H_S(Q) = H_S(P)$.

Step 3. We further prove that $H_{SR}(Q) = H_{SR}(P)$. Given a rotation group G_R, we denote its group action as $G_R(\cdot)$, then we can derive that

$$\begin{aligned} H_{SR}(P) &= G_R(H_S(P)) \\ &= \{g_j(H_S(P) | g_j \in G_R\} \\ &= \{g_j(H_S(Q) | g_j \in G_R\} \\ &= G_R(H_S(Q)) \\ &= H_{SR}(Q), \end{aligned} \tag{14}$$

where $H_{SR}(P) = G_R(H_S(P))$ holds based on the definition of rot-space in Eqn. (4) in the paper.

Step 4. Finally we prove that $\widetilde{P} = \widetilde{Q}$. As we assume the pose selector as an injective function, and we score each pose $\bar{P}_s \in H_{SR}(P)$ by different scores for different point clouds, and we select the pose $\widetilde{P} \in H_{SR}(P)$ with the largest score in $H_{SR}(P)$ which is an orderless operation. By Step 3, the rot-spaces $H_{SR}(P)$ and $H_{SR}(Q)$ contain the same set of shape poses but these shape poses are orderless. So, by our orderless pose selector, we finally select same pose, i.e., $\widetilde{P} = \widetilde{Q}$.

3 Network Details

We present the network details of pose selector Ψ, feature extractor Γ, convolution filters Φ, Θ in RFE-block in Table 1.

Table 1. Network details on pose selector Ψ, feature extractor Γ, convolution filters Φ, Θ in RFE-block. Ψ is with same parameter size in all stages. Parameter sizes in initial stage and stages 1-3 are respectively listed separated by semicolons. Numbers in each bracket represent the numbers of hidden units in successive hidden layers.

	Architecture	Parameters size	
		Classification	Segmentation
Ψ	MLP1→Max-pooling →MLP2→Softmax	MLP1:(32,64) MLP2:(128,200)	MLP1:(32,64) MLP2:(128,200)
Γ	MLP→Max-pooling	−;(64,64,64); (128,128,128);(256,256,256)	−;(128,128,128); (256,256,256);(512,512,512)
Φ	MLP	(32,32);(64,64); (128,128);(256,256)	(128,128);(128,128); (256,256);(512,512)
Θ	Matrix	$K\times32\times32;K\times64\times64;$ $K\times128\times128;K\times256\times256$	$K\times128\times128;K\times128\times128;$ $K\times256\times256;K\times512\times512$

Table 2. Shape segmentation IoU on ShapeNet part (in %) in SO(3)/SO(3) mode. "S" denotes rotation-**s**ensitive method, "R" denotes rotation-**r**obust method.

	Method	aero	bag	cap	car	chair	earph.	guitar	knife	lamp	laptop	motor	mug	pistol	rocket	skate	table
S	PointNet	**81.6**	68.7	74.0	70.3	87.6	68.5	**88.9**	80.0	74.9	83.6	**56.5**	77.6	75.2	**53.9**	**69.4**	79.9
	PointNet++	79.5	71.6	**87.7**	**70.7**	**88.8**	64.9	88.8	78.1	79.2	94.9	54.3	**92.0**	76.4	50.3	68.4	81.0
	PointCNN	78.0	**80.1**	78.2	68.2	81.2	70.2	82.0	70.6	68.9	80.8	48.6	77.3	63.2	50.6	63.2	**82.0**
	DGCNN	77.7	71.8	77.7	55.2	87.3	**68.7**	88.7	**85.5**	**81.8**	81.3	36.2	86.0	77.3	51.6	65.3	80.2
	SpiderCNN	74.3	72.4	72.6	58.4	82.0	68.5	87.8	81.3	71.3	**94.5**	45.7	88.1	**83.4**	50.5	60.8	78.3
R	RIConvNet	80.6	80.2	70.7	68.8	86.8	70.4	87.2	84.3	78.0	80.1	57.3	91.2	71.3	52.1	66.6	78.5
	Proposed	**81.5**	**80.3**	**80.0**	**74.3**	**88.7**	**72.3**	**89.9**	**85.4**	**83.2**	**88.5**	**63.5**	**92.0**	**77.8**	**56.6**	**74.4**	**81.4**

Table 3. Shape segmentation IoU on ShapeNet part (in %) in z/SO(3) mode. "S" denotes rotation-**s**ensitive method, "R" denotes rotation-**r**obust method.

	Method	aero	bag	cap	car	chair	earph.	guitar	knife	lamp	laptop	motor	mug	pistol	rocket	skate	table
S	PointNet	40.4	48.1	46.3	24.5	45.1	**39.4**	29.2	42.6	52.7	36.7	**21.2**	55.0	29.7	26.6	32.1	35.8
	PointNet++	**51.3**	**66.0**	50.8	**25.2**	**66.7**	27.7	29.7	**65.6**	**59.7**	70.1	17.2	**67.3**	**49.9**	23.4	**43.8**	**57.6**
	PointCNN	21.8	52.0	**52.1**	23.6	29.4	18.2	**40.7**	36.9	51.1	33.1	18.9	48.0	23.0	27.7	38.6	39.9
	DGCNN	37.0	50.2	38.5	24.1	43.9	32.3	23.7	48.6	54.8	28.7	17.8	74.4	25.2	24.1	43.1	32.3
	SpiderCNN	48.8	47.9	41.0	25.1	59.8	23.0	28.5	49.5	45.0	**83.6**	20.9	55.1	41.7	**36.5**	39.2	41.2
R	RIConvNet	80.6	80.0	70.8	68.8	86.8	70.3	87.3	84.7	77.8	80.6	57.4	91.2	71.5	52.3	66.5	78.4
	Proposed	**81.5**	**80.3**	**80.0**	**74.3**	**88.7**	**72.3**	**89.9**	**85.4**	**83.2**	**88.5**	**63.5**	**92.0**	**77.8**	**56.6**	**74.4**	**81.4**

4 Per-class IoU on ShapeNet Part dataset

In Tables 2, 3, we present per-class IoU scores on ShapeNet part dataset in SO(3)/SO(3) and z/SO(3) mode respectively. Our PR-invNet achieves significant higher accuracies than rotation-robust RIConvNet method in both modes. PR-invNet achieves the highest scores in z/SO(3) mode compared with all the compared rotation-sensitive and rotation-robust methods.