Supplementary Material for Structural Triangulation: A Closed-Form Solution to Constrained 3D Human Pose Estimation

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1 Detailed Inductions, Proofs and Expressions

In this section, we provide detailed inductions, proofs of the property and the lemma, and expressions of some important constants in the main paper. Note that the equation numbers are continued on the main paper, so the numbers begin from 17 in this material, and smaller numbers refer to equations proposed in the main paper.

1.1 Detailed Expression of Eq. (7)

In Section 3.2 of the main paper, we mention that if all bone vectors ($\boldsymbol{b}$) are known, then the optimal root joint position ($\boldsymbol{x}_0$) can be represented in form of

$$\boldsymbol{x}_0 = \boldsymbol{Q}\boldsymbol{b} + \boldsymbol{p}.$$

where $\boldsymbol{Q} \in \mathbb{R}^{3\times 3n}$ and $\boldsymbol{p} \in \mathbb{R}^3$ are both constants.

To validate this, we start from the formulated optimization problem, i.e.,

$$\min_{\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_n} \quad g(\boldsymbol{x}) = \sum_{i=0}^{n}\sum_{k=1}^{c} w'_{i,k} \left\| \boldsymbol{P}_k^u H(\boldsymbol{x}_i) - \hat{\boldsymbol{x}}_{i,k}\left(\boldsymbol{p}_k^\top H(\boldsymbol{x}_i)\right)\right\|^2, \tag{17}$$

$$\text{s.t.} \quad \|\boldsymbol{x}_i - \boldsymbol{x}_j\| = L_j, \forall (i,j) \in \mathcal{B}. \tag{18}$$

If all bone vectors are fixed, then constraints in Eq. (18) are already satisfied, the only problem need to be considered is how to solve Eq. (17) without constraints. Since $w'_{i,k}$ is treated as constant, the problem is a trivial unconstrained quadratic optimization problem. The conclusion is obviously correct, and now we want to provide the analytical expressions of $\boldsymbol{Q}$ and $\boldsymbol{p}$ by detailed inductions.

Before starting, we need to first regulate some notations of camera parameters. For the k^{th} camera, use $\boldsymbol{R}_k \in \mathbb{R}^{3\times 3}$ and $\boldsymbol{t}_k \in \mathbb{R}^3$ to represent the rotation and translation, and $\boldsymbol{K}_k \in \mathbb{R}^{3\times 3}$ for the intrinsic matrix. Then the projection matrix is:

$$\boldsymbol{P}_k = \left[\boldsymbol{K}_k\boldsymbol{R}_k \mid -\boldsymbol{K}_k\boldsymbol{R}_k\boldsymbol{t}_k\right]. \tag{19}$$

Split $\boldsymbol{P}_k$ by row, and also by column in the way like (19), which produces:

$$\boldsymbol{P}_k = \begin{bmatrix} \boldsymbol{P}_k^u \\ \boldsymbol{p}_k^\top \end{bmatrix} = \begin{bmatrix} \boldsymbol{p}_{1,k}^{u\top} \\ \boldsymbol{p}_{2,k}^{u\top} \\ \boldsymbol{p}_k^\top \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_{1,k}^\top \; t_{1,k} \\ \boldsymbol{r}_{2,k}^\top \; t_{2,k} \\ \boldsymbol{r}_{3,k}^\top \; t_{3,k} \end{bmatrix}, \tag{20}$$

where $\boldsymbol{K}_k\boldsymbol{R}_k = [\boldsymbol{r}_{1,k}, \boldsymbol{r}_{2,k}, \boldsymbol{r}_{3,k}]^\top$ and $-\boldsymbol{K}_k\boldsymbol{R}_k\boldsymbol{t}_k = [t_{1,k}, t_{2,k}, t_{3,k}]^\top$.

Let $\hat{\boldsymbol{x}}_{i,k} = [u_{i,k}, v_{i,k}]^\top$, then

$$\begin{aligned} \left\| \boldsymbol{P}_k^u H(\boldsymbol{x}_i) - \hat{\boldsymbol{x}}_{i,k} \left(\boldsymbol{p}_k^\top H(\boldsymbol{x}_i) \right) \right\|^2 = &\left((\boldsymbol{p}_{1,k}^u - u_{i,k}\boldsymbol{p}_k)^\top H(\boldsymbol{x}_i) \right)^2 \\ &+ \left((\boldsymbol{p}_{2,k}^u - v_{i,k}\boldsymbol{p}_k)^\top H(\boldsymbol{x}_i) \right)^2. \end{aligned} \tag{21}$$

And the first term on the right side can be reformulated as

$$\begin{aligned} (\boldsymbol{p}_{1,k}^u - u_{i,k}\boldsymbol{p}_k)^\top H(\boldsymbol{x}_i) &= (\boldsymbol{r}_{1,k} - u_{i,k}\boldsymbol{r}_{3,k})^\top \boldsymbol{x}_i + t_{1,k} - u_{i,k}t_{3,k} && (22) \\ &= \begin{bmatrix} 1 \; 0 \; -u_{i,k} \end{bmatrix} \boldsymbol{K}_k\boldsymbol{R}_k\boldsymbol{x}_i - \begin{bmatrix} 1 \; 0 \; -u_{i,k} \end{bmatrix} \boldsymbol{K}_k\boldsymbol{R}_k\boldsymbol{t}_k && (23) \\ &= \begin{bmatrix} 1 \; 0 \; -u_{i,k} \end{bmatrix} \boldsymbol{K}_k\boldsymbol{R}_k \left(\boldsymbol{x}_i - \boldsymbol{t}_k \right). && (24) \end{aligned}$$

Likewise,

$$(\boldsymbol{p}_{2,k}^u - v_{i,k}\boldsymbol{p}_k)^\top \tilde{\boldsymbol{x}}_i = \begin{bmatrix} 0 \; 1 \; -v_{i,k} \end{bmatrix} \boldsymbol{K}_k\boldsymbol{R}_k \left(\boldsymbol{x}_i - \boldsymbol{t}_k \right) \tag{25}$$

Then, $\boldsymbol{g}(\boldsymbol{x})$ can be reformulated as follows:

$$\begin{aligned} g(\boldsymbol{x}) &= \sum_{i=0}^{n}\sum_{k=1}^{c} w'_{i,k} \left(((\boldsymbol{p}_{1,k}^u - u_{i,k}\boldsymbol{p}_k)^\top \tilde{\boldsymbol{x}}_i)^2 + ((\boldsymbol{p}_{2,k}^u - v_{i,k}\boldsymbol{p}_k)^\top \tilde{\boldsymbol{x}}_i)^2 \right) && (26) \\ &= \frac{1}{2}\sum_{i=0}^{n}\sum_{k=1}^{c} 2w'_{i,k} \Big(\left(\boldsymbol{x}_i - \boldsymbol{t}_k \right)^\top \boldsymbol{R}_k^\top \boldsymbol{K}_k^\top \\ &\qquad \begin{bmatrix} 1 & 0 & -u_{i,k} \\ 0 & 1 & -v_{i,k} \\ -u_{i,k} & -v_{i,k} & u_{i,k}^2 + v_{i,k}^2 \end{bmatrix} \boldsymbol{K}_k\boldsymbol{R}_k \left(\boldsymbol{x}_i - \boldsymbol{t}_k \right) \Big) && (27) \\ &\stackrel{\text{def}}{=} \frac{1}{2}\sum_{i=0}^{n}\sum_{k=1}^{c} \left(\boldsymbol{x}_i - \boldsymbol{t}_k \right)^\top \boldsymbol{M}_{i,k}(\boldsymbol{x}_i - \boldsymbol{t}_k) && (28) \\ &\stackrel{\text{def}}{=} \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{D}\boldsymbol{x} - \boldsymbol{m}^\top \boldsymbol{x} + s. && (29) \end{aligned}$$

Eq. (29) is of standard quadratic form, and the above definitions are listed below:

$$\boldsymbol{M}_{i,k} = 2w'_{i,k}\boldsymbol{R}_k^\top \boldsymbol{K}_k^\top \begin{bmatrix} 1 & 0 & -u_{i,k} \\ 0 & 1 & -v_{i,k} \\ -u_{i,k} & -v_{i,k} & u_{i,k}^2 + v_{i,k}^2 \end{bmatrix} \boldsymbol{K}_k \boldsymbol{R}_k \tag{30}$$

$$\boldsymbol{D} = \operatorname{diag}\left\{\sum_{k=1}^{c} \boldsymbol{M}_{0,k}, \sum_{k=1}^{c} \boldsymbol{M}_{1,k}, \dots, \sum_{k=1}^{c} \boldsymbol{M}_{n,k}\right\} \tag{31}$$

$$\boldsymbol{m} = \left[\left(\sum_{k=1}^{c} \boldsymbol{M}_{0,k}\boldsymbol{t}_k\right)^\top, \left(\sum_{k=1}^{c} \boldsymbol{M}_{1,k}\boldsymbol{t}_k\right)^\top, \dots, \left(\sum_{k=1}^{c} \boldsymbol{M}_{n,k}\boldsymbol{t}_k\right)^\top\right]^\top \tag{32}$$

$$s = \frac{1}{2}\sum_{i=0}^{n}\sum_{k=1}^{c} \boldsymbol{t}_k^\top \boldsymbol{M}_{i,k}\boldsymbol{t}_k \tag{33}$$

Back to the problem we discussed in the beginning: solving the optimal root position $\boldsymbol{x}_0$ with respect to given bone vectors $\boldsymbol{b}$. With Eq. (6), we can write an objective function $p(\boldsymbol{x}_0)$ which equals $\boldsymbol{g}(\boldsymbol{x})$ so that the only unknown $\boldsymbol{x}_0$ is isolated.

$$p(\boldsymbol{x}_0) = g\left(\mathcal{G}^{-1}\begin{bmatrix}\boldsymbol{x}_0 \\ \boldsymbol{b}\end{bmatrix}\right) = \frac{1}{2}\left[\boldsymbol{x}_0^\top \ \boldsymbol{b}^\top\right]\mathcal{G}^{-\top}\boldsymbol{D}\mathcal{G}^{-1}\begin{bmatrix}\boldsymbol{x}_0 \\ \boldsymbol{b}\end{bmatrix} - \boldsymbol{m}^\top\mathcal{G}^{-1}\begin{bmatrix}\boldsymbol{x}_0 \\ \boldsymbol{b}\end{bmatrix} + s \tag{34}$$

where $\mathcal{G}^{-\top} = (\mathcal{G}^{-1})^\top$.

The first order necessary condition to minimize $p(\boldsymbol{x}_0)$ produces

$$\nabla_{\boldsymbol{x}_0} p(\boldsymbol{x}_0) = \begin{bmatrix}\boldsymbol{I}_3 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}_{3n\times 3n}\end{bmatrix}\mathcal{G}^{-\top}\left(\boldsymbol{D}\mathcal{G}^{-1}\begin{bmatrix}\boldsymbol{x}_0 \\ \boldsymbol{b}\end{bmatrix} - \boldsymbol{m}\right) = 0, \tag{35}$$

where $\boldsymbol{0}_{p\times q} \in \mathbb{R}^{p\times q}$ means a matrix of $p \times q$ whose elements all equal 0.

By definition, $\mathcal{G}^{-1}$ is of the following form

$$\mathcal{G}^{-1} = \begin{bmatrix}\boldsymbol{I}_3 & \boldsymbol{0}_{3\times n} \\ \boldsymbol{I}_{row}^{(n)\top} & \mathcal{F}\end{bmatrix} \tag{36}$$

where $\boldsymbol{I}_{row}^{(m)} = [\boldsymbol{I}_3, \boldsymbol{I}_3, \cdots, \boldsymbol{I}_3]$ (repeated m times), and $\mathcal{F} \in \mathbb{R}^{3n\times 3n}$ just represents the above part of $\mathcal{G}^{-1}$. Obviously,

$$\begin{bmatrix}\boldsymbol{I}_3 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}_{3n\times 3n}\end{bmatrix}\mathcal{G}^{-\top} = \boldsymbol{I}_{row}^{(n+1)}. \tag{37}$$

Since $\boldsymbol{D}$ is block diagonal, we have

$$\boldsymbol{I}_{row}^{(n+1)}\boldsymbol{D} = \left[\sum_{k=1}^{c} \boldsymbol{M}_{0,k}, \sum_{k=1}^{c} \boldsymbol{M}_{1,k}, \cdots, \sum_{k=1}^{c} \boldsymbol{M}_{n,k}\right] \stackrel{\text{def}}{=} \boldsymbol{M}_{row}. \tag{38}$$

Then we can induct from Eq. (35) that

$$\boldsymbol{M}_{row}\mathcal{G}^{-1}\begin{bmatrix}\boldsymbol{x}_0\\ \boldsymbol{b}\end{bmatrix} - \boldsymbol{I}_{row}^{(n+1)}\boldsymbol{m} = \boldsymbol{0}. \tag{39}$$

Let $\boldsymbol{M}_S = \sum_{i=0}^{n}\sum_{k=1}^{c}\boldsymbol{M}_{i,k}$, then

$$\boldsymbol{M}_S\boldsymbol{x}_0 + \boldsymbol{M}_{row}\mathcal{F}\boldsymbol{b} - \boldsymbol{I}_{row}^{(n+1)}\boldsymbol{m} = 0. \tag{40}$$

Therefore,

$$\boldsymbol{x}_0 = -\boldsymbol{M}_S^{-1}\boldsymbol{M}_{row}\mathcal{F}\boldsymbol{b} + \boldsymbol{M}_S^{-1}\boldsymbol{I}_{row}^{(n+1)}\boldsymbol{m}. \tag{41}$$

So the expressions of constants in Eq. (7) are

$$\boldsymbol{Q} = -\boldsymbol{M}_S^{-1}\boldsymbol{M}_{row}\mathcal{F}, \tag{42}$$

$$\boldsymbol{p} = \boldsymbol{M}_S^{-1}\boldsymbol{I}_{row}^{(n+1)}\boldsymbol{m}. \tag{43}$$

1.2 Proof of Property 1

In this section we provide detailed proof and concrete expressions of constants in Property 1.

Property 1. The optimization problem in Eq. (1) and (2) is formulated as

$$\min_{\boldsymbol{b}} \quad f(\boldsymbol{b}) = \frac{1}{2}\boldsymbol{b}^\top\boldsymbol{A}\boldsymbol{b} - \boldsymbol{\beta}^\top\boldsymbol{b} + d \tag{44}$$

$$\text{s.t.} \quad h_i(\boldsymbol{b}_i) = \|\boldsymbol{b}_i\|^2 = L_i^2,\ i = 1, 2, \ldots, n. \tag{45}$$

where $\boldsymbol{A} \in \mathbb{R}^{3n\times 3n}$ is a symmetric positive semi-definite constant matrix, $\boldsymbol{\beta} \in \mathbb{R}^{3n}$ and $d \in \mathbb{R}$ are constants. $\boldsymbol{A}$ is singular if and only if $\exists i = 0, 1, \ldots, n$, there holds $\forall k_1, k_2 = 1, 2, \ldots, c, l_{i,k_1} /\!/ l_{i,k_2}$.

Proof. First, the objective function is reformulated as Eq. (17), along with some approximations. And from the previous section, we see that Eq. (17) can be further formulated as Eq. (29), which is the beginning of this proof.

Now we need to convert joint position vector $\boldsymbol{x}$ to bone vector $\boldsymbol{b}$, involving Eq. (6) and Eq. (7).

Define $\tilde{\boldsymbol{Q}} = [\boldsymbol{Q}^\top, \boldsymbol{I}_n]^\top$ and $\tilde{\boldsymbol{p}} = [-\boldsymbol{p}^\top, \boldsymbol{0}_{1\times 3n}]^\top$, then

$$f(\boldsymbol{b}) = g\left(\mathcal{G}^{-1}\begin{bmatrix}\boldsymbol{x}_0\\ \boldsymbol{b}\end{bmatrix}\right) = g\left(\tilde{\boldsymbol{Q}}\boldsymbol{b} - \tilde{\boldsymbol{p}}\right) \tag{46}$$

$$= \frac{1}{2}(\tilde{\boldsymbol{Q}}\boldsymbol{b} - \tilde{\boldsymbol{p}})^\top\mathcal{G}^{-\top}\boldsymbol{D}\mathcal{G}^{-1}(\tilde{\boldsymbol{Q}}\boldsymbol{b} - \tilde{\boldsymbol{p}}) - \boldsymbol{m}^\top\mathcal{G}^{-1}(\tilde{\boldsymbol{Q}}\boldsymbol{b} - \tilde{\boldsymbol{p}}) + s \tag{47}$$

$$\stackrel{\text{def}}{=} \frac{1}{2}\boldsymbol{b}^\top A\boldsymbol{b} - \boldsymbol{\beta}^\top\boldsymbol{b} + d. \tag{48}$$

It is the form in (44) where

$$\boldsymbol{A} = \tilde{\boldsymbol{Q}}^\top \mathcal{G}^{-\top} \boldsymbol{D} \mathcal{G}^{-1} \tilde{\boldsymbol{Q}}; \tag{49}$$

$$\boldsymbol{\beta} = \tilde{\boldsymbol{Q}}^\top \mathcal{G}^{-\top} \boldsymbol{D} \mathcal{G}^{-1} \tilde{\boldsymbol{p}} + \tilde{\boldsymbol{Q}}^\top \mathcal{G}^{-1} \boldsymbol{m}; \tag{50}$$

$$d = s + \frac{1}{2} \tilde{\boldsymbol{p}}^\top \mathcal{G}^{-\top} \boldsymbol{D} \mathcal{G}^{-1} \tilde{\boldsymbol{p}} + \boldsymbol{m}^\top \mathcal{G}^{-1} \tilde{\boldsymbol{p}}. \tag{51}$$

And the constraints in Eq. (2) are trivially equivalent to Eq. (45). Therefore, the formulation is done.

Now we want to prove that $\boldsymbol{A}$ is symmetric positive semi-definite, and that the condition when $\boldsymbol{A}$ becomes singular is as stated in Property 1. In the following proof, "$\succeq 0$" means positive semi-definite.

We can conclude directly from definition Eq. (49) that $\boldsymbol{A}$ is congruent to $\boldsymbol{D}$. Therefore the key is to make sure $\boldsymbol{D}$ is symmetric and positive semi-definite. Since $\boldsymbol{D}$ is block diagonal, it is sufficient to prove each block matrix on its diagon has that property. We can even prove a stronger proposition, i.e., $\forall i = 0, 1, \ldots, n; k = 1, 2, \ldots, c$, s.t. $\boldsymbol{M}_{i,k}^\top = \boldsymbol{M}_{i,k}$ and $\boldsymbol{M}_{i,k} \succeq 0$.

In (30), apparently $\boldsymbol{M}_{i,k}$ is congruent to

$$\boldsymbol{N}_{i,k} = \begin{bmatrix} 1 & 0 & -u_{i,k} \\ 0 & 1 & -v_{i,k} \\ -u_{i,k} & -v_{i,k} & u_{i,k}^2 + v_{i,k}^2 \end{bmatrix} \tag{52}$$

$\boldsymbol{N}_{i,k}$ is symmetric. The eigen values of $\boldsymbol{N}_{i,k}$ are

$$\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 1 + u_{i,k}^2 + v_{i,k}^2.$$

The corresponding unit eigen vector of λ_1 is

$$\boldsymbol{\alpha}_{i,k} = \frac{1}{\sqrt{1 + u_{i,k}^2 + v_{i,k}^2}} \begin{bmatrix} u_{i,k} \\ v_{i,k} \\ 1 \end{bmatrix}, \tag{53}$$

which is the basic solution to $\boldsymbol{x}^\top \boldsymbol{N}_{i,k} \boldsymbol{x} = 0$. And according to Eq. (30), the base solution to $\boldsymbol{x}^\top \boldsymbol{M}_{i,k} \boldsymbol{x} = 0$ is $\boldsymbol{R}_k^\top \boldsymbol{K}_k^{-1} \boldsymbol{\alpha}_{i,k}$.

All eigen values are non-negative, so $\boldsymbol{N}_{i,k} \succeq 0$. And because of congruence, there also holds that $\boldsymbol{M}_{i,k} \succeq 0$.

Consider the condition when $\boldsymbol{D}$ is singular. As mentioned before, $\boldsymbol{D} \succeq 0$, so $|\boldsymbol{D}| = 0 \Leftrightarrow \exists \boldsymbol{y} \in \mathbb{R}^{3(n+1)}, \boldsymbol{y} \neq \boldsymbol{0}$, s.t. $\boldsymbol{y}^\top \boldsymbol{D} \boldsymbol{y} = 0$. Let $\boldsymbol{y} = [\boldsymbol{y}_0^\top, \boldsymbol{y}_1^\top, \ldots, \boldsymbol{y}_n^\top]^\top$. Because $\boldsymbol{D}$ is block diagonal and positive semi-definite, the above proposition equals to: $\exists i = 0, 1, \ldots, n, \boldsymbol{y}_i \in \mathbb{R}^3$ and $\boldsymbol{y}_i \neq \boldsymbol{0}$, s.t. $\boldsymbol{y}_i^\top \sum_{k=0}^{c} \boldsymbol{M}_{i,k} \boldsymbol{y}_i = 0$. And because $\boldsymbol{M}_{i,k} \succeq 0, k = 1, 2, \ldots, c$, there holds

$$\boldsymbol{y}_i^\top \sum_{k=0}^{c} \boldsymbol{M}_{i,k} \boldsymbol{y}_i = 0 \Leftrightarrow \boldsymbol{y}_i^\top \boldsymbol{M}_{i,k} \boldsymbol{y}_i = 0, k = 1, 2, \ldots, c \tag{54}$$

$$\Leftrightarrow \boldsymbol{y}_i /\!/ \boldsymbol{R}_k^\top \boldsymbol{K}_k^{-1} \boldsymbol{\alpha}_{i,k}, k = 1, 2, \ldots, c. \tag{55}$$

According to the definition of intrinsic $\boldsymbol{K}_k$ and extrinsic $\boldsymbol{R}_k$, the expression in (55) is parallel to the line connecting camera optic center and the estimated point on image plane, which are marked as $l_{i,k}$ in the main paper. Note that i is already fixed, so we can further put the proposition in (54) equivalently as:

$$\forall k_1, k_2 = 1, 2, \ldots, c, l_{i,k_1} /\!/ l_{i,k_2}. \tag{56}$$

Because of congruence, the above condition is also the necessary and sufficient condition when $|\boldsymbol{A}| = 0$. That completes the proof.

1.3 Proof of Lemma 1

Lemma 1. *Suppose $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{A}$ is non-singular. $\|\cdot\|$ is the spectral norm of a matrix. If $\|\boldsymbol{A}^{-1}\boldsymbol{B}\| < 1$, then $\boldsymbol{A} - \boldsymbol{B}$ is non-singular and we have the following inequality:*

$$\left\|(\boldsymbol{A}-\boldsymbol{B})^{-1} - \left(\boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\right)\right\| \le \frac{\left\|\boldsymbol{A}^{-1}\boldsymbol{B}\right\|^2 \left\|\boldsymbol{A}^{-1}\right\|}{1 - \|\boldsymbol{A}^{-1}\boldsymbol{B}\|}. \tag{57}$$

Proof. We first introduce Neumann Lemma:

Lemma 2 (Neumann Lemma). *Suppose $\boldsymbol{E} \in \mathbb{R}^{n\times n}$, $\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$ stands for identity matrix. $\|\cdot\|$ is a matrix norm that satisfies $\|\boldsymbol{I}_n\| = 1$. If $\|\boldsymbol{E}\| < 1$, then $\boldsymbol{I} - \boldsymbol{E}$ is non-singular and the following equation holds:*

$$(\boldsymbol{I}_n - \boldsymbol{E})^{-1} = \sum_{k=0}^{\infty} \boldsymbol{E}^k. \tag{58}$$

Because $\boldsymbol{A}$ is non-singular, we have

$$(\boldsymbol{A}-\boldsymbol{B})^{-1} = (\boldsymbol{I}_n - \boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{A}^{-1}. \tag{59}$$

Because the spectral norm of $\boldsymbol{I}_n$ is 1, Neumann Lemma is applicable here. We can extract $(\boldsymbol{I}_n - \boldsymbol{A}^{-1}\boldsymbol{B})^{-1}$ in series form, i.e.,

$$(\boldsymbol{A}-\boldsymbol{B})^{-1} = \sum_{k=0}^{\infty}(\boldsymbol{A}^{-1}\boldsymbol{B})^k\boldsymbol{A}^{-1} \tag{60}$$

$$= \boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1} + (\boldsymbol{A}^{-1}\boldsymbol{B})^2\sum_{k=0}^{\infty}(\boldsymbol{A}^{-1}\boldsymbol{B})^k\boldsymbol{A}^{-1}. \tag{61}$$

According to the properties of matrix norm, there holds

$$\left\|(\boldsymbol{A}-\boldsymbol{B})^{-1} - (\boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1})\right\| = \left\|(\boldsymbol{A}^{-1}\boldsymbol{B})^2\sum_{k=0}^{\infty}(\boldsymbol{A}^{-1}\boldsymbol{B})^k\boldsymbol{A}^{-1}\right\| \tag{62}$$

$$\le \left\|\boldsymbol{A}^{-1}\right\| \left\|\boldsymbol{A}^{-1}\boldsymbol{B}\right\|^2 \sum_{k=0}^{\infty} \left\|\boldsymbol{A}^{-1}\boldsymbol{B}\right\|^k \tag{63}$$

$$= \frac{\left\|\boldsymbol{A}^{-1}\boldsymbol{B}\right\|^2 \left\|\boldsymbol{A}^{-1}\right\|}{1 - \|\boldsymbol{A}^{-1}\boldsymbol{B}\|}. \tag{64}$$

That completes the proof.

1.4 Detailed Process to Find Solution from KKT Condition

The KKT condition of the optimization problem in Property. 1 produces

$$\nabla_{\boldsymbol{b}} l(\boldsymbol{b}, \boldsymbol{\lambda}) = (\boldsymbol{A} + 2\boldsymbol{\Lambda})\boldsymbol{b} - \boldsymbol{\beta} = 0; \tag{65}$$

$$\nabla_{\boldsymbol{\lambda}} l(\boldsymbol{b}, \boldsymbol{\lambda}) = \boldsymbol{h}(\boldsymbol{b}) - \boldsymbol{L} = 0. \tag{66}$$

By combining Eq. (65) with the given approximation in Eq. (13), we have

$$\boldsymbol{b} = (\boldsymbol{A} + 2\boldsymbol{\Lambda})^{-1}\boldsymbol{\beta} \tag{67}$$

$$\approx (\boldsymbol{I}_n - 2\boldsymbol{A}^{-1}\boldsymbol{\Lambda})\boldsymbol{A}^{-1}\boldsymbol{\beta} \tag{68}$$

$$= (\boldsymbol{I}_n - 2\boldsymbol{A}^{-1}\boldsymbol{\Lambda})\boldsymbol{b}^{(0)}. \tag{69}$$

We can take a particular bone vector from $\boldsymbol{b}^{(0)}$ by matrix multiplication:

$$\boldsymbol{b}_i^{(0)} = [\boldsymbol{0}_{3\times 3(i-1)}, \boldsymbol{I}_3, \boldsymbol{0}_{3\times 3(n-i)}]\boldsymbol{b}^{(0)}. \tag{70}$$

Define $\boldsymbol{E}_{ii} \in \mathbb{R}^{3n\times 3n}$ as

$$\boldsymbol{E}_{ii} = \begin{bmatrix} \boldsymbol{0}_{3(i-1)\times 3(i-1)} & \boldsymbol{0}_{3(i-1)\times 3} & \boldsymbol{0}_{3(i-1)\times 3(n-i)} \\ \boldsymbol{0}_{3\times 3(i-1)} & \boldsymbol{I}_3 & \boldsymbol{0}_{3\times 3(n-i)} \\ \boldsymbol{0}_{3(n-i)\times 3(i-1)} & \boldsymbol{0}_{3(n-i)\times 3} & \boldsymbol{0}_{3(n-i)\times 3(n-i)} \end{bmatrix}, \tag{71}$$

Then,

$$\|\boldsymbol{b}_i\|^2 = \boldsymbol{b}^\top \boldsymbol{E}_{ii}\boldsymbol{b} \tag{72}$$

$$= \boldsymbol{b}^{(0)\top}(\boldsymbol{I}_n - 2\boldsymbol{\Lambda}\boldsymbol{A}^{-1})\boldsymbol{E}_{ii}(\boldsymbol{I}_n - 2\boldsymbol{A}^{-1}\boldsymbol{\Lambda})\boldsymbol{b}^{(0)} \tag{73}$$

$$= \left\|\boldsymbol{b}_i^{(0)}\right\|^2 - 4\boldsymbol{b}^{(0)\top}(\boldsymbol{E}_{ii}\boldsymbol{A}^{-1}\boldsymbol{\Lambda})\boldsymbol{b}^{(0)} + 4\boldsymbol{b}^{(0)\top}\boldsymbol{\Lambda}\boldsymbol{A}^{-1}\boldsymbol{E}_{ii}\boldsymbol{A}^{-1}\boldsymbol{\Lambda}\boldsymbol{b}^{(0)}, \tag{74}$$

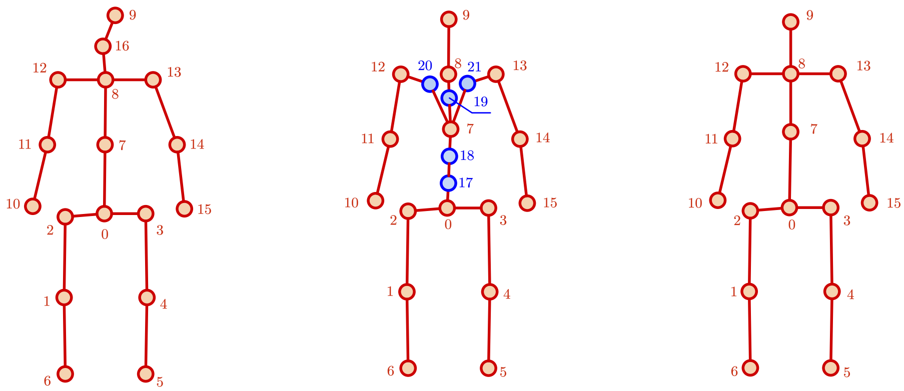
where the unknown $\boldsymbol{\lambda}$ is of second-order in the last term. If $\|2\boldsymbol{A}^{-1}\boldsymbol{\Lambda}\| \ll 1$, then the term is very small compared to the former two. So we can abandon it and derive the following equation:

$$\forall i = 1, 2, \ldots, n, \text{s.t. } 4\boldsymbol{b}^{(0)\top}(\boldsymbol{E}_{ii}\boldsymbol{A}^{-1}\boldsymbol{\Lambda})\boldsymbol{b}^{(0)} = \left\|\boldsymbol{b}_i^{(0)}\right\|^2 - \|\boldsymbol{b}_i\|^2. \tag{75}$$

Similar to $\boldsymbol{0}_{p\times q}$, $\boldsymbol{1}_{p\times q} \in \mathbb{R}^{p\times q}$ means a matrix of $p\times q$ whose elements all equal 1. Define $\boldsymbol{D}_n^{(3\times 1)} = \text{diag}\{\boldsymbol{1}_{3\times 1}, \boldsymbol{1}_{3\times 1}, \ldots, \boldsymbol{1}_{3\times 1}\}$, where the vector $\boldsymbol{1}_{3\times 1}$ repeated n times.

Then we can reformulate (75):

$$[\boldsymbol{0}_{3(i-1)}^\top, \boldsymbol{b}_i^\top, \boldsymbol{0}_{3(n-i)}^\top]\boldsymbol{A}^{-1}\text{diag}\{\boldsymbol{b}^{(0)}\}\boldsymbol{D}_n^{(3\times 1)}\boldsymbol{\lambda} = \frac{1}{4}(L_i^{(0)2} - L_i^2). \tag{76}$$



(a) 17-joint subset in Human3.6M

(b) Full 21-joint model in Total Capture

(c) 16-joint subset in Total Capture

Fig. 1: Joint indices of 3 different skeletal models. Joints with the same indices are treated as corresponding ones. Note that blue joints are ignored in our tests.

For i vary from 1 to n, we concatenate all (76) together and get

$$\boldsymbol{D}_n^{(3\times1)\top}\mathrm{diag}\{\boldsymbol{b}^{(0)}\}\boldsymbol{A}^{-1}\mathrm{diag}\{\boldsymbol{b}^{(0)}\}\boldsymbol{D}_n^{(3\times1)}\boldsymbol{\lambda} = \frac{1}{4}(\boldsymbol{L}^{(0)} - \boldsymbol{L}), \tag{77}$$

which directly produces Eq. (14) in the main paper.

2 Joint Correspondences between labels of Human3.6M and Total Capture

Fig. 1 shows the skeletal models used in two datasets, and Table. 1 shows the corresponding joint names.

3 Detailed Experiment Settings and Full Results

3.1 Lagrangian Algorithm and Relative Hyperparameter Settings

Lagrangian Algorithm for optimization problems with equality constraints is based on Lagrangian multiplier. It updates both the variable and the multiplier in a dual process. In this section, we describe the process of how our problem (Eq. (8) and (9)) is solved by Lagrangian Algorithm and how the hyperparameters are set. Detailed introduction of the algorithm can be found in [1].

Consistent with the main paper, we use $\boldsymbol{\lambda}$ to represent the multiplier and Eq. (10) for the Lagrangian function. Use $\boldsymbol{b}^{(i)}$ and $\boldsymbol{\lambda}^{(i)}$ for the value in ith iteration, and $i = 0$ refers to the initial value. Then we can write the update process as:

Table 1: Joint names and correspondences, where "Index" stands for joint indices in Fig. 1, "H36M" means Human3.6M dataset while "TC" means Total Capture Dataset. The numbers inside parentheses refer to the joint numbers in corresponding skeletal models.

Index	H36M (17)	TC (21)	TC (16)	Index	H36M (17)	TC (21)	TC (16)
0	Hips	Hips	Hips	11	RElbow	RightForeArm	RightForeArm
1	RKnee	RightLeg	RightLeg	12	RShoulder	RightArm	RightArm
2	RHip	RightUpLeg	RightUpLeg	13	LShoulder	LeftArm	LeftArm
3	LHip	LeftUpLeg	LeftUpLeg	14	LElbow	LeftForeArm	LeftForeArm
4	LKnee	LeftLeg	LeftLeg	15	LWrist	LeftHand	LeftHand
5	LFoot	LeftFoot	LeftFoot	16	Neck	\	\
6	RFoot	RightFoot	RightFoot	17	\	Spine1	\
7	Spine	Spine2	Spine2	18	\	Spine2	\
8	Thorax	Neck	Neck	19	\	Spine3	\
9	Head	Head	Head	20	\	RightSholder	\
10	RWrist	RightHand	RightHand	21	\	LeftSholder	\

$$\boldsymbol{b}^{(k+1)} = \boldsymbol{b}^{(k)} - \alpha \left(\nabla f(\boldsymbol{b}^{(k)}) + \nabla \boldsymbol{h}(\boldsymbol{b}^{(k)})^{\top} \boldsymbol{\lambda}^{(k)} \right), \tag{78}$$

$$\boldsymbol{\lambda}^{(k+1)} = \boldsymbol{\lambda}^{(k)} + \gamma \left(\boldsymbol{h}(\boldsymbol{b}^{(k)}) - \boldsymbol{L} \right). \tag{79}$$

Combining with Eq. (9) and Eq. (10), we can reformulate Eq. (78) to:

$$\boldsymbol{b}^{(k+1)} = \boldsymbol{b}^{(k)} - \alpha \left(\boldsymbol{A}\boldsymbol{b}^{(k)} - \boldsymbol{\beta} + 2\mathrm{diag}\{\boldsymbol{b}^{(k)}\} \boldsymbol{D}_n^{(3\times1)} \boldsymbol{\lambda}^{(k)} \right) \tag{80}$$

Then, in each iteration, we update $\boldsymbol{b}$ by Eq. (80) and then $\boldsymbol{\lambda}$ by Eq. (79). When iteration number gets the predefined upper limit N_{iter}, the process is terminated and $\boldsymbol{b}^{(N_{\mathrm{iter}})}$ is returned as the optimal solution.

In our experiment, the hyperparameters are: $\alpha = 2 \times 10^{-9}, \gamma = 0.5$ and $N_{\mathrm{iter}} = 50$.

3.2 Full Results of Absolute MPJPE on Human3.6M Dataset

The results of absolute MPJPE error in subjects are reported in Table 2.

3.3 Camera Parameters to Synthesize Data

When synthesizing data, all joint positions are measured in mm. The intrinsic matrix $\boldsymbol{K}$ and extrinsic translation $\boldsymbol{t}$ of all cameras are:

$$\boldsymbol{K} = \begin{bmatrix} 900 & 0.5 & 500 \\ 0 & 900 & 500 \\ 0 & 0 & 1 \end{bmatrix}; \quad \boldsymbol{t} = \begin{bmatrix} 0 \\ 0 \\ 2000 \end{bmatrix}$$

Table 2: Full table of absolute MPJPE (mm) on Human3.6M Dataset. "AT" and "VT" are short for Algobraic Triangulation and Volumetric Triangulation. Results in our method are highlighted in light gray and "*" means the estimated bone lengths are used.

Method	Dire.	Disc.	Eat	Greet	Phone	Photo	Pose	Purch.
AT by Iskakov[2]	18.06	20.56	17.70	17.06	18.75	19.30	17.50	19.12
VT by Iskakov[2]	17.08	**18.39**	16.92	16.49	**17.53**	18.29	16.52	17.81
Ours* (SCA)	18.81	20.01	16.44	17.26	18.68	18.59	17.52	18.67
Ours (w/o SCA)	16.58	18.75	15.50	16.29	17.96	18.03	16.56	17.88
Ours (SCA)	**16.43**	18.58	**15.48**	**16.13**	17.79	**17.82**	**16.47**	**17.44**
Method	Sit	SitD	Smoke	Wait	Walk	WalkD	WalkT	Avg
AT by Iskakov[2]	**20.07**	22.69	19.59	17.72	18.12	20.68	17.98	19.26
VT by Iskakov[2]	**20.06**	**20.45**	18.34	17.02	17.27	19.02	**16.81**	17.93
Ours* (SCA)	20.86	21.43	18.92	17.85	18.90	19.87	18.48	18.90
Ours (w/o SCA)	20.68	21.29	18.36	17.55	17.54	18.81	17.31	17.98
Ours (SCA)	**20.06**	20.63	**18.19**	**16.74**	**17.52**	**18.67**	17.34	**17.78**

3.4 Full Experiment Results on Synthesized Data

In this section, we report the full experiment result on synthesized data in Tables 3˜8. These tables are named in the form "criterion (unit) + camera setup". Note that σ is the standard deviation of Gaussian noise and c is the number of cameras.

References

1. Chong, E.K., Zak, S.H.: An introduction to optimization. John Wiley & Sons (2004)
2. Iskakov, K., Burkov, E., Lempitsky, V., Malkov, Y.: Learnable triangulation of human pose. In: ICCV (2019)

Table 3: Absolute MPJPE (mm) of Baseline + Round

σ (px) \ c	2	3	4	5	6	7	8	9	10
2	15.05	5.58	4.41	3.90	3.52	3.26	3.04	2.87	2.73
4	45.19	11.22	8.85	7.82	7.06	6.54	6.11	5.75	5.49
6	72.60	16.88	13.24	11.72	10.59	9.78	9.16	8.67	8.22
8	110.32	22.35	17.71	15.65	14.10	13.09	12.21	11.58	10.98
10	121.36	27.99	22.15	19.62	17.75	16.42	15.36	14.47	13.75
12	166.04	33.64	26.58	23.65	21.33	19.72	18.44	17.43	16.58
14	220.95	39.10	31.15	27.48	24.89	22.99	21.67	20.44	19.45
16	222.95	44.86	35.61	31.66	28.66	26.58	24.85	23.47	22.37
18	269.95	50.44	40.13	35.57	32.16	30.10	28.01	26.48	25.24
20	395.90	56.37	44.64	39.82	36.11	33.47	31.35	29.65	28.34

Table 4: Absolute MPJPE (mm) of Ours + Round

σ (px) \ c	2	3	4	5	6	7	8	9	10
2	13.64	4.43	3.50	3.14	2.82	2.62	2.44	2.30	2.20
4	28.00	8.89	7.01	6.29	5.67	5.26	4.90	4.62	4.40
6	43.25	13.34	10.51	9.40	8.46	7.87	7.34	6.95	6.61
8	60.89	17.65	14.01	12.52	11.30	10.48	9.80	9.26	8.75
10	79.34	22.00	17.50	15.67	14.11	13.18	12.20	11.52	10.94
12	91.80	26.55	20.95	18.87	17.01	15.71	14.65	13.87	13.16
14	105.14	30.76	24.48	21.80	19.80	18.24	17.20	16.17	15.35
16	115.35	35.33	27.89	25.14	22.59	21.02	19.54	18.44	17.58
18	123.88	39.72	31.37	28.10	25.40	23.62	21.96	20.71	19.65
20	139.26	44.24	34.80	31.29	28.23	26.11	24.48	23.03	22.02

Table 5: Absolute MPJPE (mm) of Baseline + Half

σ (px) \ c	2	3	4	5	6	7	8	9	10
2	8.45	5.91	4.83	4.17	3.75	3.43	3.19	2.98	2.82
4	16.86	11.86	9.63	8.39	7.48	6.92	6.36	5.98	5.62
6	25.25	17.73	14.51	12.58	11.30	10.32	9.57	8.93	8.45
8	33.82	23.67	19.33	16.72	15.05	13.81	12.81	12.01	11.28
10	42.29	29.63	24.23	21.01	18.88	17.28	16.06	14.99	14.21
12	50.50	35.59	29.16	25.25	22.71	20.79	19.36	18.14	17.06
14	59.21	41.53	33.88	29.55	26.51	24.31	22.63	21.34	20.17
16	67.58	47.50	38.94	33.94	30.37	27.93	26.04	24.35	23.17
18	76.17	53.56	43.97	38.44	34.43	31.64	29.42	27.60	26.08
20	84.48	59.56	48.82	42.67	38.27	35.32	32.81	30.86	29.07

Table 6: Absolute MPJPE (mm) of Ours + Half

σ (px) \ c	2	3	4	5	6	7	8	9	10
2	6.40	4.69	3.87	3.36	3.02	2.77	2.58	2.40	2.28
4	12.77	9.42	7.76	6.75	6.05	5.58	5.14	4.82	4.54
6	19.15	14.04	11.61	10.12	9.10	8.31	7.71	7.19	6.82
8	25.85	18.73	15.43	13.46	12.07	11.13	10.30	9.64	9.08
10	32.57	23.55	19.32	16.79	15.16	13.86	12.90	12.02	11.38
12	38.93	28.23	23.25	20.22	18.14	16.60	15.49	14.52	13.59
14	45.88	32.82	26.95	23.50	21.12	19.40	18.04	16.95	16.04
16	52.58	37.56	30.95	27.04	24.13	22.15	20.69	19.30	18.32
18	59.39	42.45	34.84	30.43	27.22	24.95	23.25	21.80	20.40
20	65.16	46.87	38.64	33.80	30.22	27.71	25.63	24.23	22.74

Table 7: Rate of cases our method outperforms Baseline (%) + Half

σ (px) \ c	2	3	4	5	6	7	8	9	10
2	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
4	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
6	99.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
8	99.5	100.0	100.0	99.9	100.0	100.0	100.0	100.0	100.0
10	99.3	99.9	100.0	100.0	100.0	100.0	99.9	99.9	99.9
12	99.0	99.9	99.9	99.9	100.0	99.9	100.0	99.8	100.0
14	98.5	99.4	99.9	99.8	99.9	99.8	99.9	99.8	99.8
16	99.2	99.5	99.6	99.7	100.0	99.9	99.7	99.9	100.0
18	98.0	99.0	99.4	99.6	99.8	99.7	99.8	99.9	99.7
20	98.4	98.8	99.5	99.4	99.8	99.6	99.9	99.6	99.9

Table 8: Rate of cases our method outperforms Baseline (%) + Round

σ (px) \ c	2	3	4	5	6	7	8	9	10
2	93.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
4	90.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
6	87.3	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
8	85.1	99.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0
10	82.7	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
12	82.6	99.6	100.0	100.0	100.0	100.0	100.0	100.0	100.0
14	83.4	99.4	100.0	100.0	100.0	100.0	100.0	100.0	100.0
16	85.1	98.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0
18	84.9	98.8	99.9	99.9	100.0	100.0	100.0	100.0	100.0
20	85.6	99.1	100.0	99.8	100.0	100.0	100.0	100.0	100.0