A Direct Approach to Viewing Graph Solvability Supplementary Material

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In this supplementary document we report explicit formulas for the derivatives of our polynomial equations with respect to their unknowns, that are the basis of the Jacobian check employed by our approach. In addition, we summarize the comparison with [1] and [5] in terms of size of the respective matrices.

A Useful Facts

The derivatives of functions involving vectors and matrices ultimately lead back to the partial derivatives of the individual components, and it is all about how to arrange these partial derivatives. There are several conventions, we follow [4].

Definition 1. *Let $\mathcal{F} : \mathbb{R}^{n \times q} \to \mathbb{R}^{m \times p}$ be a differentiable function. The derivative of $\mathcal{F}$ in X is the matrix $mp \times nq$:*

$$\mathsf{D}\,\mathcal{F}(X) = \frac{\partial \operatorname{vec} \mathcal{F}(X)}{\partial (\operatorname{vec} X)^\top}. \tag{1}$$

By this definition, if $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^m$ then $\mathsf{D}\,\mathcal{F}(\mathbf{x})$ coincides with the usual *Jacobian matrix* of $\mathcal{F}$. With respect to other ways of arranging partial derivatives, this definition allows to apply the chain rule. The following theorem is very useful, as it transforms the problem of finding the Jacobian of a matrix function, into the problem of finding its differential, which is usually easier.

Theorem 1 (Identification). *Let $\mathcal{F} : \mathbb{R}^{n \times q} \to \mathbb{R}^{m \times p}$ be a differentiable function and let* d *be the differential. Then*

$$\mathrm{d} \operatorname{vec} \mathcal{F}(X) = A(X)\, \mathrm{d} \operatorname{vec} X \tag{2}$$

is equivalent to

$$\mathsf{D}\,\mathcal{F}(X) = A(X). \tag{3}$$

As an example, one can easily find out the derivative of the matrix function

$$\mathcal{F}(X) = AXB$$

where A, X and B be matrices of sizes $m \times n$, $n \times p$ and $p \times q$, respectively. In fact, by the so-called "vectorization trick" [3], which states that $\operatorname{vec}(AXB) = (B^\top \otimes A) \operatorname{vec} X$, it follows that:

$$\mathrm{d} \operatorname{vec}(\mathcal{F}(X)) = \mathrm{d} \operatorname{vec}(AXB) = (B^\top \otimes A)\, \mathrm{d} \operatorname{vec} X. \tag{4}$$

Therefore, by the identification theorem we get:

$$\mathsf{D}(AXB) = (B^\top \otimes A). \tag{5}$$

From this, it follows as a special case:

$$\begin{aligned} \mathsf{D}(AX) &= (I_p \otimes A) \\ \mathsf{D}(XB) &= (B^\top \otimes I_n). \end{aligned} \tag{6}$$

Furthermore, it can be also shown that [4]:

$$\mathsf{D}(X^\top) = K_{n,p} \tag{7}$$

where $K_{n,p}$ is the *commutation matrix*, namely the $np \times np$ matrix such that $\operatorname{vec}(A) = K_{n,p} \operatorname{vec}(A^\top)$ for any A of size $n \times p$. Moreover, for A and B of sizes $m \times n$ and $p \times q$ respectively, we have [4]:

$$B \otimes A = K_{m,p}(A \otimes B)K_{q,n}. \tag{8}$$

The commutation matrix is a permutation, hence it is orthogonal: $K_{n,p}K_{n,p}^\top = I_{nq}$. Please note also that $K_{n,p} = K_{p,n}^\top$.

B Derivatives of Our Polynomial Equations

First, we are going to compute the Jacobian of the matrix function

$$\mathcal{F}(P_i, P_j) = S + S^\top \tag{9}$$

with respect to the cameras P_j and P_i, where F_{ij} is their fundamental matrix and $S = P_j^\top F_{ij} P_i$. It is clear that derivatives with respect to other cameras are zero as only two cameras are involved. Using formulas from Sec. A, we get:

$$\frac{\partial \operatorname{vec} S}{\partial (\operatorname{vec} P_j)^\top} = ((F_{ij}P_i)^\top \otimes I_4)K_{3,4} = K_{4,4}(I_4 \otimes (F_{ij}P_i)^\top) \tag{10}$$

$$\frac{\partial \operatorname{vec} S}{\partial (\operatorname{vec} P_i)^\top} = I_4 \otimes (P_j^\top F_{ij}) \tag{11}$$

and

$$\frac{\partial \operatorname{vec} S^\top}{\partial (\operatorname{vec} P_j)^\top} = I_4 \otimes (F_{ij}P_i)^\top \tag{12}$$

$$\frac{\partial \operatorname{vec} S^\top}{\partial (\operatorname{vec} P_i)^\top} = K_{4,4}(I_4 \otimes (P_j^\top F_{ij})) \tag{13}$$

Hence:

$$\frac{\partial \operatorname{vec} \mathcal{F}}{\partial (\operatorname{vec} P_j)^\top} = (K_{4,4} + I_{16}) \left(I_4 \otimes (F_{ij}P_i)^\top\right) \tag{14}$$

$$\frac{\partial \operatorname{vec} \mathcal{F}}{\partial (\operatorname{vec} P_i)^\top} = (K_{4,4} + I_{16}) \left(I_4 \otimes (P_j^\top F_{ij})\right). \tag{15}$$

The Jacobians (14) and (15) are the blocks that constitute the full Jacobian matrix. They have 12 columns each and 16 rows but only 10 corresponding to the lower (or upper) triangular part of $S + S^\top$ are used, because of symmetry. The full Jacobian has a block structure that follows the incidence matrix B of the view graph, which has one row for every edge and one column for every node. In the row of B that represents the edge (i, j) there is a -1 in column i and a $+1$ in column j and other entries are zero; in the full Jacobian the $+1$ is replaced by the 10×12 block (14) and the -1 is replaces by the 10×12 block (15):

$$\left[0 \cdots 0, \frac{\partial \operatorname{vec} \mathcal{F}}{\partial (\operatorname{vec} P_i)^\top}, 0 \cdots 0, \frac{\partial \operatorname{vec} \mathcal{F}}{\partial (\operatorname{vec} P_j)^\top}, 0 \cdots 0\right]. \tag{16}$$

Matrices of the form (16) are then stacked for all the edges in the graph.

The additional equation that fixes the rank of a camera P is:

$$z \det(W) + 1 = 0 \tag{17}$$

where z is an auxiliary variable and

$$W = \begin{bmatrix} P \\ \mathbf{a} \end{bmatrix} \tag{18}$$

with a random constant $\mathbf{a}$. It is easy to see that, thanks to the Laplace expansion of the determinant, we have:

$$\frac{\partial z \det(W)}{\partial [W]_{i,j}} = z(-1)^{i+j} \det(W^{(i,j)}) \tag{19}$$

where $[W]_{i,j}$ is the (i, j) entry of W and $W^{(i,j)}$ is the 3×3 matrix obtained removing row i and column j from W. Since the last row of W is constant we compute (19) for $i = 1 \dots 3$ and $j = 1 \dots 4$ only, obtaining a vector of 12 derivatives with respect to the entries of P. The derivative with respect to the auxiliary variable z is trivial:

$$\frac{\partial z \det(W)}{\partial z} = \det(W). \tag{20}$$

As for the derivatives of the other additional equations, the ones that fix scales and projective ambiguity are constant matrices of zero and ones.

C Number of Rows/Columns for Different Formulations

Trager et al. [5]. The solvability matrix of [5] is made of blocks, where each block comprises 20 equations, and the number of blocks per node is $d_i(d_i - 1)/2$, where d_i denotes the degree of node i (see Tab. 2 in [1]). By summing over all the nodes in the graph, we get that the number of rows of the solvability matrix (or, equivalently, the number of equations) can be calculated as:

$$e_1 = 10 \sum_{i=1}^{n} (d_i^2 - d_i) + 15 + m = 10 \sum_{i=1}^{n} d_i^2 - 19m + 15 \tag{21}$$

where $15 + m$ accounts for the additional equations introduced to remove the ambiguities and $\sum_{i=1}^{n} d_i = 2m$ due to the *degree sum formula* [2]. Recall that m denotes the number of edges. By the Cauchy-Schwarz inequality[4] we obtain:

$$\sum_{i=1}^{n} d_i^2 \geq \frac{1}{n} \left(\sum_{i=1}^{n} d_i \right)^2 \tag{22}$$

hence, using again the degree sum formula, e_1 can be bounded as

$$e_1 \geq \frac{10}{n} (2m)^2 - 19m + 15 = 40\frac{m^2}{n} - 19m + 15. \tag{23}$$

Hence the number of rows grows asymptotically (at least) as $O(n^3)$ for a dense graph (i.e., $m = O(n^2)$) and $O(n)$ for a sparse one (i.e., $m = O(n)$).

Arrigoni et al. [1]. The reduced solvability matrix used by [1] is made of blocks of 11 equations, where the number of blocks per node is $d_i - 1$ (therefore it scales linearly in the degree of node i whereas in [5] the growth is quadratic). Hence the number of rows is given by:

$$e_2 = 11 \sum_{i=1}^{n} (d_i - 1) + 15 + m = 11 \sum_{i=1}^{n} d_i - 11n + 15 + m = 23m - 11n + 15 \tag{24}$$

Therefore the number of rows of the reduced solvability matrix grows asymptotically as $O(n^2)$ for a dense graph and $O(n)$ for a sparse one. Away from the limit case of a perfectly sparse graph with $m = O(n)$, there is an advantage of this formulation with respect to [5]. In concrete terms, it is enough that $m > n$ to ensure that $e_2 \leq e_1$: indeed, after proper simplifications, (23) $\geq$ (24) becomes $40m^2 + 11n^2 \geq 42nm > 42n^2$, which reduces to $40m^2 > 31n^2$, which is always satisfied under the hypothesis $m > n$. The number of columns (i.e., variables) is the same for [5] and [1], and it is given by $v_1 = v_2 = 16m$.

Our Formulation. As explained in the main paper, our polynomial system employs a total of $e_3 = 10m + 2n + 14$ equations and $v_3 = 13n - 1$ unknowns. Hence, the number of rows of our Jacobian matrix grows asymptotically as $O(n^2)$ for a dense graph and $O(n)$ for a sparse one. In concrete terms, however, $e_3 \leq e_2$ as soon as $m \geq n - \frac{1}{13}$, i.e., $m > n$ (being m and n integers).

[4] $\left(\sum_{i=1}^{n} x_i y_i \right)^2 \leq \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right)$.

A summary is reported in Tab. 1. Note that our formulation is the only one where the number of columns scales with the number of nodes (instead of edges) in the graph, as ours is the first node-based method, as explained in the main paper. Observe also that practical datasets are far from the sparse graph approximation, as the number of edges is much larger than the number of nodes.

Table 1: Number of equations and number of unknowns for the three formulations. The row counts are in decreasing order as soon as $m > n$.

Method	#rows	#cols
Trager et al. [5]	$\geq 40m^2/n - 19m + 15$	$16m$
Arrigoni et al. [1]	$23m - 11n + 15$	$16m$
Ours	$10m + 2n + 14$	$13n - 1$

References

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